The Big, Bigger, and Biggest Five of Reverse Mathematics

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We provide an overview of our recent joint work on the Reverse Mathematics (RM for short) of the uncountable ([3–9]).

The well-known *Big Five phenomenon* of RM (see [2, 10]) is the observation that a large number of theorems from ordinary mathematics are either provable in the base theory of RM or equivalent to one of only four systems; these five systems together are called the 'Big Five' of RM. The aim of this paper is to **greatly** extend the Big Five phenomenon, working in Kohlenbach's *higher-order* RM ([1]).

In particular, we have established numerous equivalences involving the **second-order** Big Five systems on one hand, and well-known **third-order** theorems from analysis about (possibly) discontinuous functions on the other hand. We study both relatively tame notions, like cadlag or Baire 1, and potentially wild ones, like quasi-continuity.

We also show that *slight* generalisations and variations (involving e.g. the notions Baire 2 and cliquishness) of the aforementioned third-order theorems fall *far* outside of the Big Five. These observation give rise to four new 'Big' third-order systems that boast many equivalences, namely the *uncountability of* $\mathbb{R}$ ([5, 7, 9]), the *Jordan decomposition theorem* ([4, 9]), the *Baire category theorem* ([3, 8]), and Tao's *pigeon hole principle for measure* ([8, 9]).

Finally, we indicate connections to Kleene's higher-order computability theory when relevant.

References

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