Finitism distilled

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Hilbert's Program aimed to reduce the consistency of mathematics to finitistic means, identified at the time as PR mathematics. As some proposed to admit broader forms of recurrence [6, 18, 19] Tait argued agains such extensions due to their impredative aspects [15, 16, 14]. A problem with Tait's reasoninng is that already the recurrence schema itself refers to functions over $\mathbb{N}$ as completed totalities. We buttress here Tait's Thesis on grounds that avoid altogether any trace of concrete infinities or impredicativity.

In [7] we presented a formal theory of finite structures, and proved that it is mutually interpretable with Peano Arithmetic. We modify that theory here by restricting it to finitistically meaningful formulas. We further show that this theory matches a natural finitistic programming language [8].

Building blocks. Posit a denumerable set A of *atoms.* A *(k-ary) fp-function* is a partial function $F : A^* \rightharpoonup A$. An *fp-structure over a vocabulary* V is a mapping σ assigning to each $\mathbf{f} \in V$ an fp-function $\sigma(\mathbf{f})$, A finite set of fp-structures is easily representable as an fp-structure.

A-terms, denoting atoms and *F-terms,* denoting fp-functions, are generated simultaneously: $\boldsymbol{\omega}$ and A-variables are A-terms, $\boldsymbol{\emptyset}_k$ and k-ary function-variables are k-ary F-terms; an application of an F-term to A-terms yields an A-term; and an F-term $\mathbf{F}$ can be "extended" to $\{\vec{\mathbf{t}} \mapsto \mathbf{q}\}\,\mathbf{F}$ where $\vec{\mathbf{t}}, \mathbf{q}$ are A-terms.

The *formulas* of the language $\mathcal{L}_{FP}$ are generated from equations $\mathbf{t} \simeq \mathbf{q}$ between A-terms using connectives and quantifiers over A-variables and F-variables. A formula without F-quantifiers is *elementary.* The ***concrete*** formulas are those with no positive occurrence of $\forall$-F or negative occurrence of $\exists$-F; these are the finitistically meaningful formulas.

We construe inductive data, such as elements of a free algebra, as fp-structures. For example, binary strings are fp-structures over the vocabulary with a constant `e` and unary function identifiers `0` and `1`.

Axiomatizing finitistic mathematics. The theory **ST** [7] is mutually interpretable with Peano Arithmetic, whence not finitistic. We refer here to a finitistic sub-theory $\mathbf{FST}_0$ of **FST** whose main principles are **Explicit-definition** under Zermelo's separation, an induction rule for concrete formulas (induction axiom for concrete formulas is not concrete!). and a finitistic choice principle for concrete formulas.

Finitistic programming. The programming language **STV** for transformation of fp-structures [8] is finitistic in that the iterative construct is bounded by the size of the fp-structures present, generalizing the loop programs of [9] for primitive recursive functions over $\mathbb{N}$.

The three basic structure-revisions considered are *extension* of an ff with a defined entry, *contraction* by an entry, and *inception* of a new binding of a constant id to a random fresh atom. The ***programs*** are built from revisions by relational-composition, and branching under a quantifier-free guard, and iteration under a quantifier-guard and a variant. Here a variant is a finite set of ffs, and the semantics triggers a loop re-entry only if the variant decreases (via an excess of depletions over extensions).

Theorem. *For each free algebra $\mathbb{A}$, the collection of* **STV***-programs compute all functions over $\mathbb{A}$ defined by PR.*

In fact, every **ST** *program that runs in PR time can be mapped into an extensionally equivalent* **STV** *program.*

Provable termination, finitistically. Defining function provability is unproblematic for computing over inductive data, but is less obvious when referring to more general fp-structures.

Given a vocabulary $V = \{\mathbf{f}_1 \dots \mathbf{f}_k\}$, write $\vec{g}^{\,V}$ for a vector of function-variables $g_1 \dots g_k$ with $\mathfrak{r}(g_i) = \mathfrak{r}(\mathbf{f}_i)$. Let $\mathfrak{C}$ and $\mathfrak{D}$ be classes of accessible fp-structures over vocabularies V and W, respectively. We say that a partial-mapping $\Phi : \mathfrak{C} \rightharpoonup \mathfrak{D}$ is *defined by* a formula $\psi[\vec{\mathbf{f}}^{V}; \vec{\mathbf{g}}^{W}]$ if for all accessible fp-structures $\sigma = (\vec{f})$ over V, $\psi[\vec{f}^{V}, \vec{g}^{W}]$ holds for given accessible V-structure $\sigma = (\vec{f})$ and accessible W-structure $\sigma' = (\vec{g})$ iff $\sigma' = \Phi(\sigma)$.

A mapping $\Phi : \mathfrak{C} \rightharpoonup \mathfrak{D}$ is *provably defined (in* $\mathbf{FST}_0$*) by* $\psi[\vec{\mathbf{f}}^{V}, \vec{\mathbf{g}}^{W}]$ if, in addition to the above, $\mathbf{FST}_0$ is concrete, and there are concrete formulas $\varphi_C[\vec{f}^{V}]$ and $\varphi_D[\vec{g}^{W}]$ defining $\mathcal{C}$ and $\mathcal{D}$ respectively, such that the following formulas are provable in $\mathbf{FST}_0$.

$$\varphi_C[\vec{f}] \;\rightarrow\; \exists \vec{g}\, \psi[\vec{f}, \vec{g}] \;\wedge\; \varphi_D[\vec{g}]$$

and

$$(\varphi_C[\vec{f}] \;\wedge\; \psi[\vec{f}, \vec{g}] \;\wedge\; \psi[\vec{f}, \vec{h}]) \;\rightarrow\; \vec{g} = \vec{h}$$

Finitistic programs are finitistically provable. *If a mapping Φ as above is computed by an* **STV** *program then it is provable in* $\mathbf{FST}_0$*.*

For the converse implication, we use structural induction on cut-free sequential proofs for $\mathbf{FST}_0$ to prove

Theorem. *If a mapping between fp-structures is provable in* $\mathbf{FST}_0$*, then it is computed by a program in* **STV***.*

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