An algebraic representation of information types

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This paper is strongly related to inquisitive semantics [2, 3], which is a framework for a formal representation of questions. The main idea of inquisitive semantics can be formulated in this way: in contrast to statements that convey concrete pieces of information (information tokens), questions are associated with types of information. For example, for some concrete objects a, b, one can consider the following pieces of information: *a is a circle*, *b is a triangle*, *a is red*, *b is blue*. These information tokens can be viewed as falling respectively under the following information types: *the shape of a*, *the shape of b*, *the colour of a*, *the colour of b*. Of course, there can be various tokens of the same type. For example, *a is a circle* and *a is a triangle* are both tokens of the type *the shape of a*. In inquisitive semantics, questions like *what is the shape of a* are identified with the corresponding types of information (see [1], for more details). Interestingly, one can observe that types of information can be combined by logical connectives, just like information tokens. For instance, one can form the type *the colour of a and the shape of a*, which includes, for example, the information token *a is red and a is a triangle*. This observation indicates that one can construct a language of information types and determine a logic for this language. Inquisitive logic does not use for this purpose a type theoretical language but rather employs a standard looking language of predicate logic. In this paper we will present a novel algebraic approach to the semantics of information types for the following first-order language L:

$$\varphi, \psi ::= Pt_1 \dots t_n \mid \bot \mid \varphi \to \psi \mid \varphi \wedge \psi \mid \varphi \mathbin{\vee\!\!\!\vee} \psi \mid \forall x \varphi \mid \exists x \varphi \mid \circ\varphi$$

Negation is defined in the usual way: $\neg\varphi =_{def} \varphi \to \bot$. The operators $\mathbin{\vee\!\!\!\vee}$ and $\exists$ are known as the *inquisitive disjunction* and *inquistive existential quantifier*. They can be viewed as operators that form information types from information tokens or they can represent question forming operators as in inquisitve semantics. We included into the language also an operator $\circ$ that is interpreted as follows: $\circ\varphi$ is a proposition that says that some token of the type φ holds. This allows us to define the ordinary disjunction and existential quantifier: $\varphi \vee \psi =_{def} \circ(\varphi \mathbin{\vee\!\!\!\vee} \psi)$, $\exists x\varphi =_{def} \circ\exists x\varphi$.

A novelty of our approach is that we will define the semantics with the help of the notion of an *inquisitive nucleus*. A *nucleus* on a Heyting algebra $\mathcal{H} = \langle H, \sqcap, \sqcup, \Rightarrow, 0\rangle$ is a function j satisfying the following conditions: (a) $s \sqsubseteq j(s)$, (b) $j(j(s)) \sqsubseteq j(s)$, (c) $j(s \sqcap t) = j(s) \sqcap j(t)$. A nucleus is *dense* if $j(0) = 0$. A *nuclear complete Heyting algebra* (*ncHA*) is a complete Heyting algebra equipped with a nucleus. If $\mathcal{H}$ is an ncHA, the set of all its j-fixed points, i.e. the set $jH = \{j(s) \mid s \in H\}$, will be called *the core* of $\mathcal{H}$. The elements of the core will be viewed as *information tokens* (importantly, they also form a complete Heyting algebra).

An important class of ncHAs related to the Kripke style semantics for inquisitive logic are the algebras of non-empty downward closed sets in a given complete Heyting algebra $\mathcal{H}$, with the nucleus j defined as $j(X) = \{s \in H \mid s \sqsubseteq \bigsqcup X \text{ in } \mathcal{H}\}$, for any non-empty downward closed set X in $\mathcal{H}$. We will call these structures *Kripkean ncHAs*.

By an *algebraic frame* we will understand a pair $\mathcal{F} = \langle \mathcal{H}, U \rangle$, where $\mathcal{H}$ is an ncHA and U is a non-empty set (the domain of quantification). We can introduce in the usual way valuation in the algebraic frame and evaluation of variables in the domain of quantification and, in the next step, we can define algebraic semantics for the language L by interpreting its logical symbols $\bot, \to, \wedge, \vee\!\!\!\vee, \forall, \exists, \circ$ respectively by the corresponding algebraic operations $0, \Rightarrow, \sqcap, \sqcup, \bigsqcap, \bigsqcup, j$ of ncHAs. Without further restrictions this semantics determines the so-called predicate lax logic [4]. We obtain inquisitive logic by imposing some additional restrictions. First, we restrict the valuations so that every elementary formula is semantically interpreted by an information token. More precisely, such valuation is defined as a function V which assigns to any n-ary predicate P a function $V(P) : U^n \to jH$. Second, we require that the nucleus j is dense and the following two conditions are satisfied for every $s \in H$ and any collection of indexed elements $t_i, u_{ik} \in H$, where $i \in I$ and $k \in K$, for some index sets I, K:

$$\text{(a) } j(s) \Rightarrow \bigsqcup_{i\in I} t_i = \bigsqcup_{i\in I}(j(s) \Rightarrow t_i) \qquad \text{(b) } \bigsqcap_{i\in I} \bigsqcup_{k\in K} j(u_{ik}) = \bigsqcup_{f:I\to K} \bigsqcap_{i\in I} j(u_{if(i)}).$$

The condition (a) is an algebraic counterpart of a principle of inquisitive logic called *split* and the condition (b) is complete distributivity for core elements. An ncHA with a nucleus satisfying these conditions will be called *inquisitive*. I will argue that these constrains are exactly what we need to obtain a desirable interaction between the core elements and their joins in order to interpret the core elements as information tokens and their joins as information types. In particular, we obtain that in any algebraic model of this kind any formula φ can be associated with a set of core elements $\mathcal{T}(\varphi)$ such that the algebraic value of φ in that model is the join of $\mathcal{T}(\varphi)$ (intuitively, φ represents the type of information tokens from $\mathcal{T}(\varphi)$). Moreover, $\mathcal{T}(\varphi)$ can be defined recursively in a way that reflects the usual type theoretic constructions. For example, $\mathcal{T}(\varphi \wedge \psi)$ can be defined as the set of core elements that encode pairs of elements from $\mathcal{T}(\varphi)$ and $\mathcal{T}(\psi)$, moreover, $\mathcal{T}(\varphi \to \psi)$ can be defined as the set of core elements that encode in a specific way functions form $\mathcal{T}(\varphi)$ to $\mathcal{T}(\psi)$, and so on.

I will also show that these inquisitive algebraic models are strongly related to the Kripkean models in the sense of the following theorem. Let $\mathcal{H}_1$, $\mathcal{H}_2$ be ncHAs. A complete homomorphism from $\mathcal{H}_1$ to $\mathcal{H}_2$ is a function $h : H_1 \to H_2$ which preserves all the operations $\bigsqcup, \bigsqcap, \Rightarrow, j$ and 0. We say that $\mathcal{H}_2$ is a homomorphic j-image of $\mathcal{H}_1$ if h is a complete homomorphism for which it holds that $j_2 H_2 = h(j_1 H_1)$.

Theorem 1. *An ncHA is inquisitive iff it is a homomorphic j-image of a Kripkean ncHA.*

References

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