Variants of the prenex normal form game

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Ehrenfeucht–Fraïssé game (by Ehrenfeucht 1961 and Fraïssé 1954) and its variants are an invaluable tool in finite model theory for comparing structures and anayzing the expressive power of logics. In the original game, the number of game rounds is directly related to the same quantifier rank; as such, it offers a method which is robust, but also coarse in the sense that the number of logically inequivalent sentences of rank r increases rapidly, as r approaches infinity. This has led to search for games related to finer logical resources. There have been several approaches to refinement, e.g., the Immerman game by Immerman 1981, followed by a recent work by Marco Carmosino, Ronald Fagin, Neil Immerman, Phokion G. Kolaitis, Jonathan Lenchner, and Rik Sengupta 2023.

Luosto and Hella introduced the *prenex normal game* $\mathsf{PNFG}(\vec{Q}\bar{x}, \mathfrak{A}, \mathfrak{B})$ between two structures $\mathfrak{A}$ and $\mathfrak{B}$ with the same finite vocabulary in 1999. Quite recently we undertook the project again, putting the matter in a larger perspectives and adding applications, and the results were finally published in 2024. The prenex normal form game is related to the sentences with the prenex $\vec{Q}\bar{x}$ so that the Duplicator has a winning strategy in $\mathsf{PNFG}(\vec{Q}\bar{x}, \mathfrak{A}, \mathfrak{B})$ if and only if every sentence with that prefix that is true in $\mathfrak{A}$ is also true in $\mathfrak{B}$, in symbols, $\mathfrak{A} \Rightarrow_{\vec{Q}} \mathfrak{B}$. In contrast to the EF-game, where the players alternate their moves between the structures, the game $\mathsf{PNFG}(\vec{Q}\bar{x}, \mathfrak{A}, \mathfrak{B})$ is asymmetric: There are two phases; in the first phase, the players repeat a certain semantic game in $\mathfrak{A}$ multiple time, whereas in the second phase, the players play a single semantic game in $\mathfrak{B}$. The first phase may be long: the number of rounds is bounded above by the number of complete quantifier-free types with variables $\bar{x}$ plus one.

In a more recent work, we continue the study of the prenex normal form game, looking at the complexity of the formula that the quantifier prefex binds. This leads to several variants of the basic game. We present some simple applications of the variant games.