**Epsilon Modal Logics**

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In this talk, I develop a new class of modal logics I call 'Epsilon Modal Logics' related to Hilbert and Bernays' (1939) Epsilon Calculus and based on 'epsilon modalities'. Similarly to how epsilon terms in Epsilon Calculus express indefinite descriptions of objects, epsilon modalities express indefinite descriptions of related points of evaluation (worlds). Epsilon Modal Logics therefore represent an intensional generalization of Epsilon Calculus.

Syntactically, epsilon modalities consist of connectives $\langle A\rangle$, the '$\varepsilon$-modality', and $[A]$, the '$\tau$-modality', indexed by a formula $A$ of the language. Semantically, given a Kripke model $\mathcal{M}$ based on frames $\mathcal{F} = \langle \mathcal{W}, \mathsf{R}\rangle$, their interpretation is based on an arbitrary choice function $\phi$ defined over intensions $X \subseteq \mathcal{W}$ of the model:

$$\phi(X) := \begin{cases} w \in X & \textit{if } X \neq \emptyset \\ w \in \mathcal{W} & \text{otherwise} \end{cases}$$

A formula under the scope of an epsilon modality is evaluated at a related world satisfying the index of the modality (if any) through $\phi$:

$$\mathcal{M}, w, \phi \Vdash \langle A\rangle B \quad \text{iff} \quad w\mathsf{R}w'' \text{ and } \mathcal{M}, w'', \phi \Vdash B, \quad \text{for} \quad w'' = \phi(\{w' \mid w\mathsf{R}w' \text{ and } \mathcal{M}, w', \phi \Vdash A\})$$
$$\mathcal{M}, w, \phi \Vdash [A]B \quad \text{iff} \quad \text{if } w\mathsf{R}w'', \text{ then } \mathcal{M}, w'', \phi \Vdash B, \quad \text{for} \quad w'' = \phi(\{w' \mid w\mathsf{R}w' \text{ and } \mathcal{M}, w', \phi \nVdash A\})$$

The remaining semantic clauses behave as usual. In particular, it holds that $\mathcal{M}, w, \phi \Vdash \neg A$ iff $\mathcal{M}, w, \phi \nVdash A$. If epsilon terms are allowed in the metatheory, choice functions can be avoided altogether, and the choice of world can be denoted by $\varepsilon$- and $\tau$-terms, the latter defined as $\tau x\, A := \varepsilon x\, \neg A$ as usual:

$$\mathcal{M}, w, \phi \Vdash \langle A\rangle B \quad \text{iff} \quad w\mathsf{R}w'' \text{ and } \mathcal{M}, w'', \phi \Vdash B, \quad \text{for} \quad w'' = \varepsilon w'\, (w\mathsf{R}w' \text{ and } \mathcal{M}, w', \phi \Vdash A)$$
$$\mathcal{M}, w, \phi \Vdash [A]B \quad \text{iff} \quad \text{if } w\mathsf{R}w'', \text{ then } \mathcal{M}, w'', \phi \Vdash B, \quad \text{for} \quad w' = \tau w'\, (\text{if } w\mathsf{R}w', \text{ then } \mathcal{M}, w', \phi \Vdash A)$$

I will present some technical results about these logics, and compare them similar accounts by Fitting (1972) and Chan (1987). In particular, I will show axiomatizations and metatheorems of the Epsilon versions of well-known extensions of normal modal logics. The smallest normal Epsilon Modal logic is axiomatized over a Classical propositional base by adding the following axioms and rules:

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|---|---|
| wCrit | $\langle B\rangle A \to \langle A\rangle A$ |
| Def | $\langle A\rangle C \leftrightarrow \neg[\neg A]\neg C$ |
| $\circ$Dist | $[A](B \circ C) \leftrightarrow ([A]B \circ [A]C)$, for $\circ \in \{\wedge, \vee, \to\}$ |
| $\neg$Dist | $\neg[A]B \to [A]\neg B$ |
| Ext | $[A \leftrightarrow B](A \leftrightarrow B) \to ([A]C \leftrightarrow [B]C)$ |

NEC If $\vdash A$, then $\vdash [A]A$

All Epsilon Modal logics based on first-order definable classes of frames are conservative over their non-Epsilon base. Standard $\Box$ and $\Diamond$ modalities are embedded as follows:

$$\Diamond A := \langle A \rangle A \qquad\qquad \Box A := [A]A$$

The close relationship between Epsilon Modal logics and Epsilon Calculus can be shown by their mutual embeddability, obtained extending the standard translation and Fitting's (2002) modal translation of quantifiers to the epsilon case.

The embeddability result allows for an intensional interpretation and generalizations of Epsilon Calculus' applications. One of the most interesting ones in philosophy is due to Carnap (1961). Carnap used epsilon terms to provide a solution to the so-called 'problem of theoretical terms' in the context of the formal reconstruction of scientific theories. His solution consisted in explicitly defining theoretical terms by arbitrary witnesses satisfying the laws of the theory, expressed in the object language by epsilon terms. By epsilon modalities, this approach can be generalized to the explicit definition of 'theoretical contexts' of evaluations by means of the laws holding in them. Semantically, these contexts are interpreted as the related worlds arbitrarily chosen by $\phi$ in which theory laws hold (if any).

The interpretation of epsilon modalities as definitions of non-deterministic choices of worlds satisfying certain formulas is close to that of antecedents of some accounts of conditional logics. As a further application, I will investigate interpretations of epsilon modalities as antecedents, which turn out displaying connexive features (Wansing, 2023).

## References

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