Non-deterministic approach to modality

The foundational concepts of non-deterministic or quasi-extensional semantics, initially brought forth by Ivlev [Ivl88] and Rescher [Res62], were significantly advanced and formally codified in the pivotal studies by Avron [AL04, AL05, AZ11]. Avron's contribution, the introduction of Nmatrices, marks a significant generalization of the traditional approach to many-valuedneess. The main difference between an Nmatrix and a logical matrix is the interpretation of the connectives of the language. In the classical approach connectives are interpreted as functions yielding singular outcomes. Nmatrices employ multi-functions, enabling sentences to assume multiple potential values. In this framework the value of a complex formula is not necessarily uniquely determined by the values of its propositional parameters. This shift allows these semantics to be more flexible and to be a fruitful tool for studying non-classical logic.

One of the key applications of this type of semantics is to study modality. This has been started independently by [Ivl88] and [Kea81]. Ivlev was interested in non-normal modal logics i.e. logics that do not validate the rule of necessitation. He provided non-deterministic semantics for some such systems. His idea was similar to the idea of Lukasiewicz and his modal logic. He used truth-values as means of representing the modal status of a given proposition. On his account, there are four truth-values: necessary true, contingently true, contingently false, and necessary false.

Contrary to Ivlev, Kearns was interested in providing an alternative semantics for normal modal logics due to philosophical reasons. He developed a similar four-valued approach with one major difference: instead of using the non-deterministic semantics directly, he restricts the set of valuations by means of filtration. The proposed filtration method is called the level valuation. Roughly speaking, at level 0 we start with all valuations. At level 1 we get rid of those, which do not make tautologies of 0-level necessary. We call a valuation a level valuation iff it is a m-level valuation for any natural number m.

Kearns showed that the level valuation semantics is sound and complete with respect to logics **T**, **S4**, and **S5**. This has been further studied in seminal papers [OS16, CLN19]. In these papers, the authors work expanded this foundation to encompass six-valued semantics by breaking the link between two possibility and necessity. This idea was further expanded to an eight-valued framework, distinctively addressing truth, necessity, and possibility as separate entities, resulting in eight unique combinations that serve as the semantic values. According to this idea, one treats the truth, necessity and possibility of a proposition separately. So, in total they are 8 combinations which they use as truth-values. This is nicely summarized by the following table:

Table 1: Meaning of values

Value	**Status of the sentence**
$\mathtt{T}_\Diamond$	$\Box\varphi, \Diamond\varphi, \varphi$ (necessary, possible and true)
$\mathtt{T}$	$\Box\varphi, \neg\Diamond\varphi, \varphi$ (necessary, not possible and true)
$\mathtt{t}_\Diamond$	$\neg\Box\varphi, \Diamond\varphi, \varphi$ (not necessary, possible and true)
$\mathtt{t}$	$\neg\Box\varphi, \neg\Diamond\varphi, \varphi$ (not necessary, not possible and true)
$\mathtt{f}_\Diamond$	$\neg\Box\varphi, \Diamond\varphi, \neg\varphi$ (not necessary, possible and false)
$\mathtt{F}$	$\Box\varphi, \neg\Diamond\varphi, \neg\varphi$ (necessary, not possible and false)
$\mathtt{F}_\Diamond$	$\Box\varphi, \Diamond\varphi, \neg\varphi$ (necessary, possible and false)
$\mathtt{f}$	$\neg\Box\varphi, \neg\Diamond\varphi, \neg\varphi$ (not necessary, not possible and false)

In our study, we adopt an approach that diverges slightly from those of earlier works. Rather than adjusting the interpretations of various connectives to replicate established modal logics such as **K** or its well-studied extensions, our focus is on identifying axioms that ensure the framework's robustness and completeness across its possible extensions. We begin by capturing the minimal (weakest) logic within this setting, which we denote as logic **H**. According to **H**[1] the modalities hold no intrinsic meaning and essentially operate within a propositional logic framework in a propositional language expanded with modalities.

Intriguingly, while this logic does not lend itself to finitely-many valued deterministic semantics, it does have a natural eight-valued non-deterministic semantics. Our exploration further extends into the examination of this logic's extensions through the lenses of $\Box$-refinements and $\Diamond$-refinements. Intuitively, these notions capture strengthening of the semantical interpretation of either $\Box$ or $\Diamond$ by making it more deterministic.

This enables us to comprehensively map out the extensions of these logics and to provide a natural axiomatization of thereof. We provide a general recipe for finding the axioms sound and complete to a particular refinements and prove the generic soundness and completeness theorems.

References

[AL04] Arnon Avron and Iddo Lev. Non-deterministic matrices. In *Proceedings. 34th International Symposium on Multiple-Valued Logic*, pages 282–287. IEEE, 2004.

[AL05] Arnon Avron and Iddo Lev. Non-deterministic multiple-valued structures. *Journal of Logic and Computation*, 15(3):241–261, 2005.

[AZ11] Arnon Avron and Anna Zamanski. Non-deterministic semantics for logical systems. In Gabbay and Guenthner, editors, *Handbook of Philosophical Logic*, volume 16. Springer Netherlands, 2011.

[CLN19] Marcelo Coniglio, Fariñas Del Cerro Luis, and Marques Peron Newton. Modal logic with non-deterministic semantics: Part I—Propositional case. *Logic Journal of the IGPL*, 10 2019. jzz027.

[Ivl88] Yury Ivlev. A semantics for modal calculi. *Bulletin of the Section of Logic*, 17(3/4):114–121, 1988.

[Kea81] John T Kearns. Modal semantics without possible worlds. *The Journal of Symbolic Logic*, 46(1):77–86, 1981.

[OS16] Hitoshi Omori and Daniel Skurt. More modal semantics without possible worlds. *IfCoLog Journal of Logics and their Applications*, 3(5):815–845, 2016.

[Res62] Nicholas Rescher. Quasi-truth-functional systems of propositional logic. *The Journal of Symbolic Logic*, 27(1):1–10, 1962.

[1] Named after L.Humberstone.