Entailment and Containment: a Ternary and Contextual approach to Information and Topic Inclusion

The family of relevant logics [6] and containment logics [5] are unified by their common objective of finding a way out of Lewis paradoxes of implication. However, the two research projects parted their ways as competing analyses of implication. I suggest to remedy the situation by considering a ternary approach to *relevant containment logic*.

As revealed by [3], analytic implication $\twoheadrightarrow$ in Parry's containment logic PAI has a double-barreled nature: *entailment* is modeled by $\mathsf{S4}$ strict implication and *containment* is modeled via a variable inclusion syntactic filter so that

$$\vdash_{\mathsf{PAI}} \varphi \twoheadrightarrow \psi \textit{ iff } \begin{cases} \vdash_{\mathsf{S4}} \Box(\varphi \to \psi) \\ At(\psi) \subseteq At(\varphi). \end{cases}$$

However, grounding analytic implication on $\mathsf{S4}$ has the drawback of importing all the deficiencies of classical modal logic. Following [7], relevant logic can be adopted as an alternative modal logic of entailment on which a satisfactory theory of containment can piggyback ride. However, contrary to [7], I will propose a semantic framework for relevant containment logic which combines Routley-Meyer models (defining informational content $[\![\varphi]\!]$) with generalised topic models (defining topical content $(\!|\varphi|\!)$). This provides a uniform ternary account of containment, which capitalises on a key intuition of relevant logic.

Relevant containment logic combines the best of containment logics – such as the rejection of $\varphi \twoheadrightarrow \varphi \vee \psi$ on the ground of topic containment – and relevant logic – such as the rejection of Deutsch's fallacy $(\varphi \wedge \neg\varphi) \wedge \psi \twoheadrightarrow (\psi \wedge \neg\psi)$ on paraconsistent grounds. Analytic implication is conceptually analysed as the conjunction of two theses: (i) φ informationally entails ψ and (ii) φ topically contains ψ.

(i) Entailment. Routley-Meyer possible worlds semantics – or better, information states semantics – provides the required modal analysis of entailment, as per [4]. In particular, entailment $\to$ is treated as a modality with a distinctive neighborhood ternary accessibility relation R: $RsXY$ is read as "proposition X entails Y according to the informational context fixed by state s" and may be suggestively written $X \sqsubseteq_s Y$.

It is well known that Routley-Meyer semantics divides states into two sorts, so that *non-logical* states (in which theorems can fail) are distinguished from *logical* or *normal* states (in which theorems hold). At non-logical states $\sqsubseteq_s$ has no specific properties. In particular, $\sqsubseteq_s$ is not a partial ordering (e.g. it may fail to be reflexive, as needed to invalidate $\psi \to (\varphi \to \varphi)$). However, logical states $l \in L$ subscribe to tighter logical connections. For one, they support all logically valid relevant implications; for two, $\sqsubseteq_l$ *reduces to* the partial order expected from logical entailment, i.e. set-theoretic inclusion.

(ii) Containment. Recent theories of topic [1, 2] pave the way for a more general analysis of topic containment than that offered by variable inclusion. I build on such work by observing that just as one may offer a ternary analysis of contextual entailment, a ternary analysis of contextual containment seems promising. I obtain such an analysis by generalising the partial ordering relation $\preceq_s$ at issue in Fine's topic structure.

By way of analogy with relevant logic, I offer a topical analysis of *contextual containment*. Topics are always evaluated in situ, that is, in a particular situational or discursive context. The particular background discursive topic associated with a state influences the inclusions that will be upheld in that context. While the discursive topic of a logical state induces a discursive context where the standard lattice-theoretic treatment of topics are respected, the focal topic of a non-logical world can be more focused, interrupting the standard topic-theoretic expectations.

The upshot of these considerations is that containment $\supseteq$ can be analysed as a modality with a ternary accessibility relation $\preceq_s$, such that $a \preceq_s b$ may be read as "the topic a is contained in topic b according to the discoursive context fixed by s". $\preceq_s$ need not be in general a partial order due to the transformative role exerted by the discoursive context in assessing topical inclusion. However, logical contexts are understood as discoursive context which are transparent, i.e. assessment of topical inclusion is not affected by contextual modifiers. For this reason, for all logical states $l \in L$, $\preceq_l$ *turns out* to be a partial order.

The resulting framework is one in which $\varphi \twoheadrightarrow \psi$ is provably equivalent to $\varphi \to \psi \wedge \varphi \supseteq \psi$, which in turn is semantically equivalent to the following:

$$s \models \varphi \twoheadrightarrow \psi \ \textit{iff} \ \begin{cases} [\![\varphi]\!] \sqsubseteq_s [\![\psi]\!] \\ (\!|\psi|\!)_s \preceq_s (\!|\varphi|\!)_s. \end{cases}$$

$(\!|\varphi|\!)_s$, the topical content of φ at information state s, is recursively defined in such a way that every intensional connective is associated with a distinct function on topics (generalising [1, 2]).

I then show how to pair the outlined semantics with a Hilbert-style axiomatisation, which modularly adds containment principles for $\supseteq$ to the relevant axiom system characterising entailment $\to$. Finally, I show how to obtain a soundness and completeness result for the axiom system with respect to the semantics.

References

[1] Thomas Macaulay Ferguson. Subject-matter and intensional operators I: Conditional-agnostic analytic implication. *Philosophical Studies*, (180):1849–1879, 2023.

[2] Thomas Macaulay Ferguson. Subject-matter and intensional operators iii: State-sensitive subject-matter and topic sufficiency. *The Review of Symbolic Logic*, page 1–27, 2023.

[3] Kit Fine. Analytic implication. *Notre Dame Journal of Formal Logic*, 27(2):169–179, 1986.

[4] Lou Goble. Neighborhoods for entailment. *Journal of Philosophical Logic*, 32(5):483–529, 2003.

[5] William T. Parry. *Analytic Implication; Its History, Justification and Varietiess*, pages 101–118. Springer Netherlands, Dordrecht, 1989.

[6] Richard Routley, Val Plumwood, Robert K. Meyer, and Ross T. Brady. Relevant Logics and Their Rivals. Volume I. *Ridgeview*, 1982.

[7] R. Sylvan. Relevant containment logics and frame problems. *Logique Et Analyse*, 31(21):11, 1988.