Notational Variance in Substructural Logics

Camillo Fiore, UBA / CONICET

For our purposes, a *logical system* is a pair $\langle \mathcal{L}, \Rightarrow \rangle$ where $\mathcal{L}$ is a formal language and $\Rightarrow$ is a dyadic relation standing for logical consequence in that language. There are lots of logical systems in the market. Often, however, two or more of those systems differ in a merely superficial way: they differ only in the symbols for a given constant (e.g. '$\wedge$' vs. '&' for conjunction), or in the syntactic conventions (e.g. infix vs. prefix notation), or in the primitive constants (e.g. negation and conjunction vs. negation and disjunction). When two systems differ only in such a superficial way, we say that they are *notational variants* of one another.

In the last years, we have witnessed the emergence of various logics that we may call 'radically substructural'. Traditionally, investigations in substructural logics focused mostly on properties such as contraction, exchange, monotonicity or associativity (see, e.g. [13]). The logics we have in mind, in contrast, challenge the core of the Tarskian conception of logical consequence by abandoning the properties of reflexivity and/or transitivity. Systems of this sort can serve various purposes, but their most recent popularisation has to do with the non-trivial treatment of various paradoxical phenomena (see [1] for a survey).

The guiding question of this paper is: what conditions are necessary and sufficient for two logical systems to be notational variants? First, I will argue that radically substructural logics pose serious challenges to our extant answers to these questions; in short, our usual criteria of notational variance either under-generate or over-generate in the presence of these logics. Second, I will propose new criteria which overcome these challenges.

I start the paper by laying down what I take to be the standard approach to notational variance (exemplified by [10, 11, 12]). This approach relies on a central idea, namely, that two logical systems are notational variants just in case they are coextensive modulo translation. More precisely, let $\mathbf{L}_1 = \langle \mathcal{L}_1, \Rightarrow_1 \rangle$ and $\mathbf{L}_2 = \langle \mathcal{L}_2, \Rightarrow_2 \rangle$ be logical systems. The standard approach is that these systems are notational variants just in case there is pair of translations τ_1 and τ_2 such that τ_1 faithfully embeds $\mathbf{L}_1$ in $\mathbf{L}_2$, τ_2 faithfully embeds $\mathbf{L}_2$ in $\mathbf{L}_1$, and in addition the following 'inversion' requirement is fulfilled:

$$A \Leftrightarrow_1 \tau_2(\tau_1(A)) \qquad\qquad A \Leftrightarrow_2 \tau_1(\tau_2(A))$$

The purpose of this last requirement is to guarantee that the two translations are, so to speak, mutually coherent. (Typically, some additional syntactic constraints are imposed on τ_1 and τ_2—for instance, that they be the identity function for atomic formulas. But we can bypass those constraints for the purposes of this abstract.)

In the first substantive part of the paper I analyse how non-reflexive logics pose a challenge to the standard approach. I take logic **TS** (see [9]) as my test-case. I present it as the system $\langle \mathcal{L}, \Rightarrow_{\mathbf{TS}} \rangle$, where $\mathcal{L}$ is a propositional language with constants $\neg$, $\vee$ and $\wedge$, and relation $\Rightarrow_{\mathbf{TS}}$ is defined using the strong Kleene valuations. The sense in which **TS** undermines the standard approach is straightforward: its non-reflexivity will always induce a failure of the inversion condition. To illustrate, let τ_1 and τ_2 be two copies of the identity function on $\mathcal{L}$. Clearly, τ_1 and τ_2 faithfully embed **TS** into itself. However, $p \not\Rightarrow_{\mathbf{TS}} \tau_2(\tau_1(p))$. So, the standard approach under-generates: it delivers that some systems are not notational variants of themselves. After considering various alternatives, I propose to amend the standard approach by replacing the original inversion requirement by the following one:

$$\begin{array}{rcl} A, \Gamma \Rightarrow_1 C & \text{iff} & \tau_2(\tau_1(A)), \Gamma \Rightarrow_1 C \\ \Gamma \Rightarrow_1 C & \text{iff} & \Gamma \Rightarrow_1 \tau_2(\tau_1(C)) \\ A, \Gamma \Rightarrow_2 C & \text{iff} & \tau_1(\tau_2(A)), \Gamma \Rightarrow_2 C \\ \Gamma \Rightarrow_2 C & \text{iff} & \Gamma \Rightarrow_2 \tau_1(\tau_2(C)) \end{array}$$

The resulting criterion solves the problem observed, rendering **TS** a notational variant of itself. Also, it delivers the usual verdicts in the more familiar cases. Thus, it constitutes an improvement over the standard approach.

In the second substantive part of the paper I analyse how non-transitive logics pose a challenge even to this last, amended criterion. I take logic **ST** (see [5]) as my test-case. I present it as the system $\langle \mathcal{L}, \Rightarrow_{\mathbf{ST}} \rangle$, where $\mathcal{L}$ is as before and $\Rightarrow_{\mathbf{ST}}$ is also defined using the strong Kleene valuations. I also present classical logic **CL** as the system $\langle \mathcal{L}, \Rightarrow_{\mathbf{CL}} \rangle$, where $\Rightarrow_{\mathbf{CL}}$ is defined as usual using the Boolean bivaluations. The sense in which **ST** undermines our amended criterion is the following. On the one hand, **ST** and **CL** have exactly the same valid arguments; as a consequence, they are trivially declared notational variants by our criterion. On the other hand, however, **ST** supports naive, non-trivial theories of paradoxical phenomena which trivialise **CL**; this has been considered by many authors (e.g. [3, 8, 6]) as a sufficient reason to say that **ST** and **CL** are not mere notational variants. So, the amended criterion over-generates: it declares as notational variants systems that are intuitively not. I consider some possible solutions to this problem; in particular, the proposal emerging from the works of Barrio et. al. [2, 4], according to which two logical systems are notational variants just in case they are coextensive modulo translation not only at the level of arguments, but also at the level of meta-arguments of any finite level. I claim that even this refined criterion over-generates claim of notational variance. Then, I propose my alternative solution, which builds on some insights from the literature on theoretical equivalence between logical theories (see [7, 14, 15]).[1] Basically (and omitting the formal details), I claim that two logical systems are notational variants just in case, for any possible collection of *non-logical* assumptions, the theories generated from these assumptions by the two systems are coextensive modulo translation. The innovative character of the proposal is that, in addition, I do not understand the non-logical assumptions as formulas, but rather as *arguments*. This innovation is what allows to set systems **ST** and **CL** apart. I claim that in the literature we can find independent motivation for understanding non-logical assumptions in this way, and that the criterion of notational variance I propose behaves well in the more familiar cases. Thus, I conclude that the new criterion constitutes a clear improvement over both the standard approach and its amended variant.

In short, I claim that our usual criteria of notational variance are undermined by the existence non-reflexive and non-transitive systems, and provide alternative criteria that promise to overcome this problem.

[1] Do not confuse the precise notion of a logical system with the informal notion of a logical theory. Arguably, different logical systems can be presentations of the same logical theory.

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