Causal models and their generalizations

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Causation and counterfactuals (Hume)

A counterfactual $A \mathrel{\Box\!\!\rightarrow} C$ is a conditional describing how things would go if the world differed from what it actually is (in such a way that it satisfies the antecedent A).

There is a line of thought connecting causation and counterfactuals (reducing one to the other?)

Semi-apocryphal origins in Hume (1748), *An Enquiry concerning Human Understanding*:

> *We may define a cause to be an object followed by another, and where all the objects, similar to the first, are followed by objects similar to the second. Or, in other words, where, if the first object had not been, the second never had existed.*

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Causation and counterfactuals (Lewis)

Stalnaker (1968) proposed a precise semantics for counterfactuals, then generalized by Lewis (1973).

One considers possible-worlds models with a notion of similarity between worlds (*u* may be closer than *v* to a reference world *w*).

$M, w \models A \mathrel{\Box\!\!\rightarrow} C$ if all A-worlds closest to w satisfy C.

Lewis (1973b) defines the relation "event *c* causes event *e*" as transitive closure of:

<u>*e* causally depends on *c*</u> if and only if, if *c* were to occur *e* would occur; and if *c* were not to occur *e* would not occur.

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Causal models (Pearl, Glymour?)

From statistics and computer science, there arose a different approach.

A *causal model* encodes causal laws as *structural equations*:

$$Y := f(X_1, \ldots, X_n)$$

This equation says that, when variables $X_1, \ldots X_n$ are set to values $x_1, \ldots x_n$, then variable Y takes value y.

Not the contrary! This equation does not tell you how $X_1 ... X_n$ may be affected by modifying Y.

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Causal graphs

The structural equations $Y := X, Z := X + Y$ induce a graph:

$$\begin{tikzcd} & Y \arrow[dr] & \\ X \arrow[ur] \arrow[rr] & & Z \end{tikzcd}$$

$PA_Z = \{X, Y\}$ are the *parents* (or *direct causes*) of Z
$PA_Y = \{X\}$ is the unique parent of Y.

Y, Z are **endogenous**; X is **exogenous**.

Models with **acyclic** graphs are called **recursive** models. I will mostly refer to these.

If the model is recursive, the values (/probabilities) of the exogenous variables uniquely determine the values (/probabilities) of all variables in the model.

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Interventionist counterfactuals

In a causal model, we can give precise meaning to **interventionist counterfactuals**, i.e. expressions of the form:

If variables $X_1, \ldots, X_n$ were fixed to values $x_1, \ldots, x_n$,
then ψ would hold.

Formally, we write it:

$$\mathbf{X} = \mathbf{x} \mathrel{\Box\!\!\rightarrow} \psi$$

or also:

$$[\mathbf{X} = \mathbf{x}]\psi$$

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Interventions and interventionist counterfactuals

Causal setting (M, u): $\begin{cases} Z := X \\ Y := X + Z \end{cases}$ plus an assignment $u(X) = 2$.

How do we evaluate $Z = 3 \,\Box\!\!\rightarrow Y = 5$?

1) We apply the intervention $do(Z = 3)$ to M, obtaining a new model $M_{Z=3}$.

I.e., we replace the equations with $\begin{cases} Z := 3 \\ Y := X + Z \end{cases}$

2) We evaluate Y by plugging in (in the second equation) the value $Z = 3$ and the value $u(X) = 2$; we obtain $Y = 5$ (i.e.: $(M_{Z=3}, u) \models Y = 5$).

3) $(M_{Z=3}, u) \models Y = 5$, so $(M, u) \models Z = 3 \,\Box\!\!\rightarrow Y = 5$.

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Interventionist vs. Lewisian counterfactuals

Halpern (2013) devised a way to associate to each recursive model an equivalent Lewisian model.

-This shows that recursive models can be seen as special cases of Lewisian models. They satisfy an additional axiom/rule, *Reversibility*:

$$\frac{(\mathbf{X} = \mathbf{x} \land W = w) \mathrel{\Box\!\!\rightarrow} Y = y \qquad (\mathbf{X} = \mathbf{x} \land Y = y) \mathrel{\Box\!\!\rightarrow} W = w}{(\mathbf{X} = \mathbf{x} \mathrel{\Box\!\!\rightarrow} Y = y)} (\text{if } Y \neq W)$$

-On the other hand, general causal models not always have an equivalent Lewisian model. The logics are incomparable.

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Axiomatizations

- Galles and Pearl (1998) proposed a few principles (*effectiveness, composition, reversibility*...) inspired by the *potential outcome approach* in statistics.

- Halpern (2000, 2016) systematized them into complete axiom systems (for 1) the recursive case, 2) the general case, 3) an intermediate “unique solution” case.

- Briggs (2012) extended the axiomatization to right-nested counterfactuals and (more controversially) to disjunctive antecedents.

- Zhang &al. (2013, 2023) axiomatized the class(es) of models that behave similarly to Lewisian models.

- All results above assume finitely many variables and values. Ibeling & Icard (2020) / Halpern & Peters (2021) extend the axiomatization to the infinitary case.

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Limitations of causal models: contingent dependencies?

Works of Henkin (1961), Hintikka and Sandu (1989), Väänänen (2007) have lead to an interest in studying *dependencies* in logic.

The (in)dependencies considered in this context usually make sense only when describing *multiplicities* of things (*contingent, data dependencies*). They can be explained in terms of **team semantics** (Hodges 1997), an analogue of Tarskian semantics where formulas are evaluated over *sets* of assignments (**teams**). E.g.:

$T \models =(\mathbf{X}; Y)$ iff for all assignments $s, s' \in T$, $s(\mathbf{X}) = s'(\mathbf{X})$ implies $s(Y) = s'(Y)$.

(X **functionally determines** Y)

Causal models describe only one state of the variables in the system. They are then inadequate to describe data dependencies. Vice versa, a team cannot capture causal dependencies.

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Causal teams

Barbero & Sandu (2018-2020) proposed a fusion of the two types of models.

A **causal team** of endogenous variables **V** is a pair $T = (T^-, \mathcal{F})$,

where, as before, $\mathcal{F}$ encodes some functional causal laws $\mathcal{F}_Y$ for the endogenous variables;

while T^- is a set of variable assignments *consistent with the causal laws*.

Interventions on causal teams

With the causal laws in place, to each model T we can associate another model $T_{\mathbf{X}=\mathbf{x}}$ that is obtained by intervening to fix the values of $\mathbf{X}$ to $\mathbf{x}$, e.g:

T:

$$\begin{tikzcd} X \arrow[r] \arrow[rr, bend left] & Y \arrow[r] & Z \end{tikzcd}$$

X	Y	Z
0	1	1
1	2	3
1	2	3

$\rightsquigarrow$

$$\begin{tikzcd} X \arrow[rr, bend left] & Y \arrow[r] & Z \end{tikzcd}$$

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0	**1**	...
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$\rightsquigarrow$ $T_{Y=1}$:

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0	1	**1**
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assuming $\mathcal{F}_Z(X, Y) = X + Y$ and $\mathcal{F}_Y(X) = X + 1$.
Note that the law for Y is removed after intervening on Y.

We can say that a counterfactual $\mathbf{X} = \mathbf{x} \boxright \psi$ holds in T if the consequent ψ holds in the intervened model $T_{\mathbf{X}=\mathbf{x}}$.

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Generalized causal teams

There is a natural generalization (which happens to work better technically): allowing uncertainty also about the causal laws.

Barbero & Yang (2022) introduced **generalized causal teams**, which are essentially sets of causal settings, e.g.:

X	Y	Z	
2	2	4	$\mathcal{F}$
2	2	4	$\mathcal{G}$
1	3	4	$\mathcal{G}$

with equations $\mathcal{F}_Z(X) = 2 * X$ and $\mathcal{G}_Z(X, Y) = X + Y$.

More flexibility: we could e.g. use a model to say that
- we are uncertain between two causal explanations
- we only know the causal graph but not the actual laws
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Axiomatizations, again

Barbero &Yang (2022) provided a complete system (in "natural deduction style", more or less...) for interventionist counterfactuals + functional dependencies (over generalized causal teams).

The causal team case is recovered using a rule of the form

$$\begin{array}{c} \forall \mathcal{F} \in \mathbb{F}_\sigma \\ [\Phi^{\mathcal{F}}] \\ \vdots \\ \dfrac{\psi}{\psi} \end{array}$$

where $\Phi^{\mathcal{F}}$ is a formula that explicitly describes a system $\mathcal{F}$ of causal laws.

(More efficiently, one can characterize the class of causal teams with 2 unintuitive axioms, B. & Y. 2020.)

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Limitations of causal models: indeterminate interventions

- If I press *some* button on the remote, the TV turns on.
- I don't know if I set the 1st or the 3rd gear, but the car started.
- If the pressure or the temperature are increased, the container will break.

Barbero & Yang (2022) suggest that indeterminate interventions (e.g. $do(X = 1 \text{ or } X = 3)$) can be modeled by applying separately the interventions $do(X = 1)$ and $do(X = 3)$ to the model T and then taking the union $T_{X=1} \cup T_{X=3}$.

[Notice that $T_{X=x} \cup T_{Y=y}$ should be a *generalized* causal team – since $T_{X=x}$ and $T_{Y=y}$ will typically have different systems of laws.]

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Indeterminate interventions and disjunctive antecedents

Indeterminate indeterventions provide an alternative (and better justified) semantics for counterfactuals with disjunctive antecedents, e.g.

$$T \models (X = x \lor Y = y) \mathrel{\Box\!\!\rightarrow} \psi$$

is taken to mean that ψ holds after the intervention $do(X = x \text{ or } Y = y)$, i.e., it holds in the modified model $T_{X=x} \cup T_{Y=y}$.

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Limitations of causal models: conditional interventions

The literature on causal inference often uses complex inteventions of the form “set Y to value $f(X)$”.

Such an intervention is easy to perform on a causal team (assignment-by-assignment). But how can it be represented in a causal model?

Limitations of causal models: indeterministic causal laws

The usual causal models encode causal laws as *functions*. But e.g. in physics there are many indeterministic causal laws, representable as *relations* or *multivalued functions*:

- If a cannon ball is shot at a certain angle, it will fall *within a certain range*.
- If a coin is tossed, it will *either* fall on heads or tails (and not, say, stay in my pocket).
- Particles that enter a Stern-Gerlach apparatus will go either up or down.

If we intervene on a causal setting with a relational law, we immediately obtain a (relational) causal team:

TOSSING	COIN
no	in-pocket

$\underset{do(TOSSING = yes)}{\rightsquigarrow}$

TOSSING	COIN
yes	heads
yes	tails

I.e., causal teams, not causal settings, are closed under indeterministic laws.

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Recent work on the indeterministic case

In Barbero (2024) I show how to model indeterministic causal laws in a variant of causal team semantics (this involves some nontrivial decisions).

The notions of parent set, direct cause and non-dummy argument diverge in this treatment. In general:

$$PA_V \supseteq \{\text{direct causes of } V\} \supseteq \{\text{non-dummy arguments of } V\}.$$

The paper also provides complete axiomatizations, in the **general** and the **recursive** case, in a Halpern-style language $\mathcal{H}$ with counterfactuals

$$(T^-, \mathcal{F}) \models [\mathbf{X} = \mathbf{x}]\psi \text{ iff for every } s \in T^-_{\mathbf{X}=\mathbf{x}} : (\{s\}, \mathcal{F}) \models \psi$$

Importantly, the notions of **direct cause**, **exogeneity** and **endogeneity** are *definable* in this language.

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General axiomatization

Rule MP. $\dfrac{\psi \qquad \psi \to \chi}{\chi}$ Rule NEC. $\dfrac{\vdash \psi}{\vdash [\mathbf{X}{=}\mathbf{x}]\psi}$

I0. Instances of classical tautologies.

I1°. $[\mathbf{X} = \mathbf{x}]Y = y \to [\mathbf{X} = \mathbf{x}]Y \neq y'$ (when $y \neq y'$) [Uniqueness]

I2°. $[\mathbf{X} = \mathbf{x}]\bigsqcup_{y \in \mathrm{Ran}(Y)} Y = y$ [Definiteness]

I3•. $\langle \mathbf{X} = \mathbf{x} \rangle (Z = z \,\&\, \mathbf{Y} = \mathbf{y}) \to \langle \mathbf{X} = \mathbf{x}, Z = z \rangle \mathbf{Y} = \mathbf{y}$ [Weak composition]

I4°. $[\mathbf{X} = \mathbf{x}, Y = y]Y = y$ [Effectiveness]

I5•. $[\mathbf{X} = \mathbf{x}]\psi \,\&\, [\mathbf{X} = \mathbf{x}](\psi \to \chi) \to [\mathbf{X} = \mathbf{x}]\chi$ [K-axiom]

I6°. $(\langle \mathbf{X} = \mathbf{x}, V = v \rangle (Y = y \,\&\, \mathbf{Z} = \mathbf{z}) \,\&\, \langle \mathbf{X} = \mathbf{x}, Y = y \rangle (V = v \,\&\, \mathbf{Z} = \mathbf{z})) \to$
$\langle \mathbf{X} = \mathbf{x} \rangle (V = v \,\&\, Y = y \,\&\, \mathbf{Z} = \mathbf{z})$
(for $V \neq Y$, and $\mathbf{Z} = \mathrm{Dom} \setminus (\mathbf{X} \cup \{V, Y\})$) [Weak reversibility]

I7. $\mathbf{Y} = \mathbf{y} \leftrightarrow \Box \mathbf{Y} = \mathbf{y}$. [Flatness]

I8. $\varphi_{\mathrm{Exo}(Y)} \to (\langle \mathbf{W}_Y = \mathbf{w} \rangle Y = y \leftrightarrow \Diamond Y = y)$ [Exogenous variables]

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Direct cause and the recursive case

Indeterministic case: X is a direct cause of Y if, after fixing all other variables to some values, there is an intervention on X which changes the *range of possible values* of Y.

Definable in $\mathcal{H}$:

$$X \rightsquigarrow Y : \bigsqcup_{(\mathbf{z},x,y)\in \mathrm{Ran}(\mathbf{Z}XY)} \sim(\langle \mathbf{Z}X = \mathbf{z}x\rangle Y = y \leftrightarrow \langle \mathbf{Z} = \mathbf{z}\rangle Y = y)$$

where $\mathbf{Z} = \mathrm{Dom} \setminus \{X, Y\}$.

We can *define* a causal team to be **recursive** if the graph of the direct cause relation is acyclic.

An axiomatization for recursive causal teams is then obtained by adding one (canonical) axiom scheme:

R. $(X_1 \rightsquigarrow X_2 \,\&\, \cdots \,\&\, X_{n-1} \rightsquigarrow X_n) \rightarrow \sim X_n \rightsquigarrow X_1$. [Generalized recursivity]

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A difference with the deterministic recursive case: failure of (strong) composition

Galles&Pearl and Halpern gave a simpler axiomatization for the recursive case. It included a simpler version of the *Composition* principle:

$$([\mathbf{X} = \mathbf{x}]W = w \,\&\, [\mathbf{X} = \mathbf{x}]Y = y) \to [\mathbf{X} = \mathbf{x}, W = w]Y = y$$

But this is arguably NOT VALID in indeterministic models:

A	$\multimap$ C
1	heads

$\overset{do(A=1)}{\rightsquigarrow}$

A	$\multimap$ C
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Here $T \models A = 1 \,\&\, C = heads$ but $T \not\models [A = 1]C = heads$.

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Further axiomatizations (1)

My framework allows the causal laws to be *partial*, that is, there to be interventions without an outcome.

We can then say that a causal team is **total** if the causal laws are total multivalued functions, i.e., intervening on the parent set of a given variable always produces at least one outcome.

This class of models is characterized, semantically and axiomatically, by the axiom:

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Limitations of causal models: uncertainty and learning

Many concepts used in causal reasoning involve a mixture of causal and observational notions. How to model these in causal settings?

In a causal team we can define an operator corresponding to observation/learning (*selective implication*).

Def: $T \quad \rightsquigarrow \quad T^{\psi} := \{s \in T | \{s\} \models \psi\}$

- $T \models \psi \supset \chi$ iff $T^{\psi} \models \chi$.

T:

X	Y	Z
1	2	3
2	1	1
3	2	1
3	2	2

$\rightsquigarrow$

$T^{X=3}$:

X	Y	Z
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Then $T \models X = 3 \supset Y = 2$. ("After learning that $X = 3$, certainly $Y = 2$")

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Decomposing conditional probabilities

The selective implication is particularly useful in probabilistic settings.

If we replace teams with *multiteams* (multisets of assignments), we can introduce probabilistic atoms, e.g.:

- $T \models \Pr(\alpha) > \epsilon$ iff $|T^\alpha|/|T| > \epsilon$ (or T empty).

Conditional probabilistic statements become definable:

$T \models \gamma \supset \Pr(\alpha) > \epsilon$ iff $P_T(\alpha \mid \gamma) > \epsilon$ (or T^γ empty).
$T \models \gamma \supset \Pr(\alpha) > \Pr(\beta)$ iff $P_T(\alpha \mid \gamma) > P_T(\beta \mid \gamma)$ (or T^γ empty).

[NOTE: in joint works with G. Sandu and J. Virtema, we have axiomatizations and semantic classifications of such languages.]

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Probabilistic-causal expressions

Having *both* conditionals $\Box\!\!\rightarrow$ and $\supset$ allows to encompass many special notions from causal inference, e.g.:

- **Conditional *do* expressions**: $P(Y = y \mid do(X = x), Z = z) = \epsilon$

  E.g: "The probability that a patient abandons treatment, if (s)he develops side effects, is ϵ".

  is rendered as

  $$X = x \Box\!\!\rightarrow (Z = z \supset P(Y = y) = \epsilon)$$

  ("After the intervention $do(X = x)$, the probability $P(Y = y \mid Z = z)$ is ϵ").

- **Pearl's counterfactuals**: $P(Y_{X=x} = y \mid Z = z) = \epsilon$

  E.g.:"the probability that my driving time would have been t had I taken the highway ($X = 1$), given that I have taken a local road ($X = 0$), is ϵ".

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Conditioning before AND after an intervention

In our formalism we can write:

$$Z = z \supset (X = x \;\Box\!\!\rightarrow (W = w \supset P(Y = y) = \epsilon))$$

"After the intervention $do(X = x)$, the probability of $Y = y$, conditional on Z being z before the intervention and W being w after the intervention, is ϵ"

Can this be written in the traditional formalism? Probably it would be done as follows:

$$P(Y_{X=x} = y \mid W_{X=x} = w, Z = z) = \epsilon$$

How far can the usual formalism be bent??

Conditioning between multiple interventions

$$\mathbf{X} = \mathbf{x} \;\Box\!\!\rightarrow (\chi \supset (\mathbf{W} = \mathbf{w} \;\Box\!\!\rightarrow P(Y = y) = \epsilon))$$

“After two interventions, the probability of $Y = y$ in case χ holds after the first intervention is ϵ”

Can we still write this in a Pearl-style notation? Yes, but complicated.
Using logical equivalences we get:

$$(\mathbf{X} = \mathbf{x} \;\Box\!\!\rightarrow \chi) \supset \left[(\mathbf{X}' = \mathbf{x}' \wedge \mathbf{W} = \mathbf{w}) \;\Box\!\!\rightarrow Pr(Y = y) = \epsilon\right].$$

which can be converted into:

$$Pr(\mathbf{Y}_{\mathbf{X}'=\mathbf{x}' \wedge \mathbf{W}=\mathbf{w}} = \mathbf{y} \mid \chi_{\mathbf{X}=\mathbf{x}}) = \epsilon$$

Our formalism seems to be more natural.

Normal form: a different ladder of causation

Our language $\mathcal{PCO}$ can describe very complex interactions of intervention and observation $\supset, \Box\!\!\rightarrow$.

Yet, it can be proved that any $\mathcal{PCO}$ formula is equivalent to a Boolean combination of formulas of 3 forms:

1. $\gamma \supset \Pr(\alpha) \rhd t$
   (= conditional probability statement $P(\alpha \mid \gamma) \rhd t$)
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Pearl's own "ladder of causation" also includes, at level 2, the *conditional do expressions*, say $P(\alpha \mid do(\mathbf{X} = \mathbf{x}), Z = z) \rhd t$, where $Z = z$ is post-intervention conditioning.
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Another approach: causal epistemic models

With Smets, Schulz, Velazquez-Quesada and Xie, we also proposed a *modal* version of the semantics.

- In this case, besides the causal laws and team, we have a designated ("actual") world. Formulas are evaluated at that world, by usual modal semantics. It is an enriched S5 model.

- An intervention not only modifies the team, but picks a new actual world. (This works well with deterministic, acyclic laws...)

- Now selective implication works as a public announcement operator; and one can explicitly introduce epistemic operator K.

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Does something go wrong?

The causal epistemic semantics just sketched satisfies the *no learning* principle:

$$\textbf{NL}: \quad [\mathbf{X} = \mathbf{x}]K\varphi \rightarrow K[\mathbf{X} = \mathbf{x}]\varphi.$$

Nothing can be learned from interventions. This contradicts our everyday experience that, say, if we try to turn on a torch we will learn whether its battery is empty or not.

Observables

We then modified the semantics to account for the fact that some variables are *observable*.

I.e., the state of the battery is not observable, while the light emitted by the torch is.

When we evaluate an intervention, we also eliminate all worlds that disagree with the (new) actual world about the values of the observables.

This semantics correctly predicts that we will learn something by turning on the torch.

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Thought experiments vs. real experiments

Our diagnosis:

- The semantics without observables describes *thought experiments*, in which the outcomes of interventions can be predicted “from the armchair”.

- The semantics with observables describes real-life experiments, which allow us to learn from observing the effects of our actions.

Thank you!

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