Finitism distilled

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  - Required in formalization (calculus)
  - Essential to generality of mathematics (Poincaré)
  - Central in abstractions (functionals...)

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- BUT not without reservation:
    - Infinity is at the core of paradoxes
    - The universe is likely finite
    - Spacial continuity is a deception
- Infinity is essential, finiteness remains fundamental.

Hilbert's Program

- Response to the Foundations Crisis:
- Trying to have it both ways.
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    actual infinite and all, is at least consistent
- Smashed by Gödel: Consistency unprovable, let alone using just finitistic means
- A MIRROR PROJECT (Tait):
    Delineate the finitistic core of mathematics
    without recourse to the infinite.

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  What's that?

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- $\boxed{\textit{Recurrence}}$ over $\mathbb{N}$ (Skolem 1932):
  Define $f : \mathbb{N} \times X \to X$ from g_0 and g_s.

$$\boxed{\begin{array}{rcl} f(0, X) & = & g_0(X) \\ f(\mathrm{s}(n), X) & = & g_s(f(n, X), X, n) \end{array}}$$

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- + defined using $X = \mathbb{N}$ from 0 and successor.
  Using $X = N \times (N \to N)$ yields Ackermann's function
- **PR functions:** Generated from constructors using recurrence for $X = \mathbb{N}^k$ and explicit definitions.

Primitive Recursive Arithmetic

- A formal mathematical theory for the PR world.
- Symbols and defining equations for all PR functions.
- Induction template:
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- Tait (2002,2012): φ should be predicative:
  whereas $\forall$ in φ presupposes $\mathbb{N}$ as totality.
- ***Tait's Thesis:*** **Finitistic Mathematics is precisely PRA.**

Articulating finitism explicitly

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- Looking closer, even formal PRA is not finitistic:
  it refers to PR functions, i.e. infinite objects.
- Something more is needed to justify Tait's Thesis.
- If finitism should stand on its own feet,
  it should be built up, not down.
- Focusing on finite sets impedes basic data-changes.
  We take ***finite functions (ff's)*** of arity 0,1,2 over ***"atoms."***

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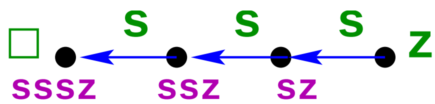
  A $\boxed{V\textit{-molecule}}$ is a W-FDS for some $W \subseteq V$.
- The $\boxed{\textit{universe}}$ of a FDS is
  the union of the domains and co-domains of its ff's.

Denotations

- $\boxed{V\text{-terms}}$ are generated from V as usual.
  A reserved id ω denotes "undefined".
  Terms without it are $\boxed{\textit{standard.}}$
- A standard V-term t has a $\boxed{\textit{value}}$ t_σ in a V-molecule σ.

Examples: Molecules for numbers and strings

Natural number 3: 

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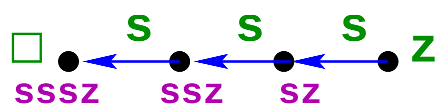
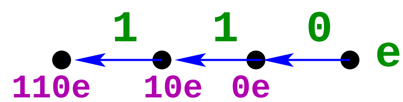


String 110



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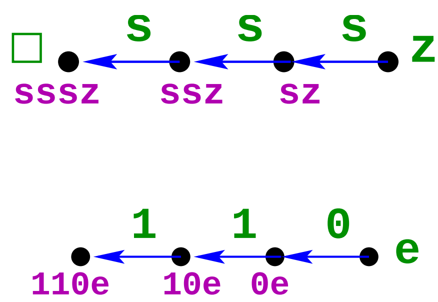

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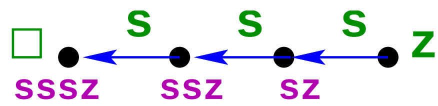

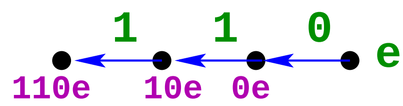

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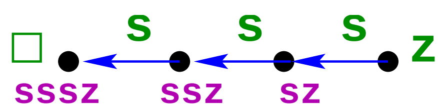

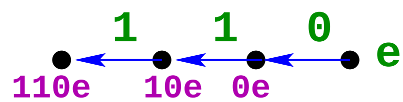

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  Considering the union as two molecules is optional.
- A second box would require a fresh id.

Computing with FDSs

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- A generic basic operation.
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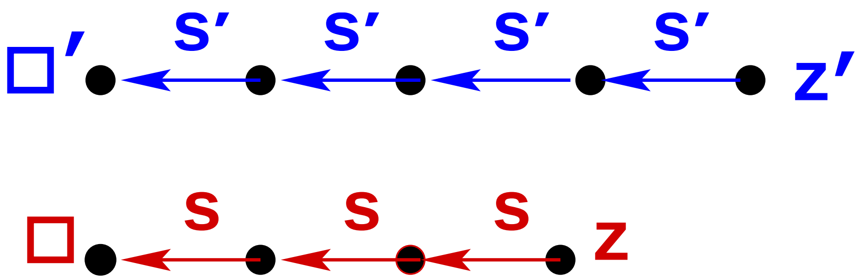

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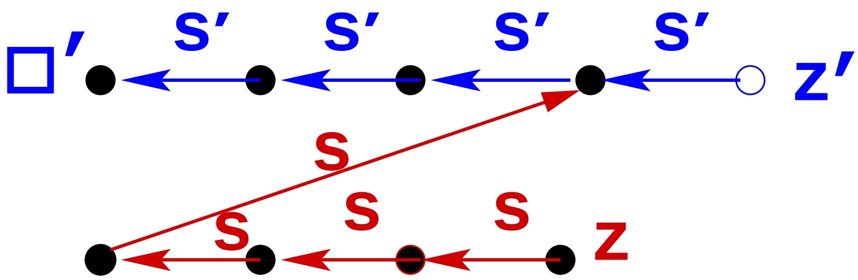

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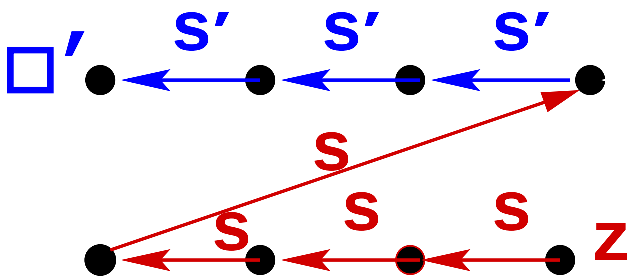

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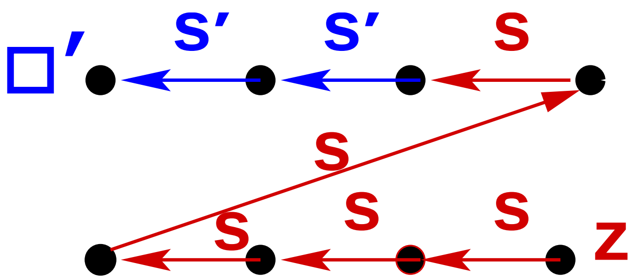

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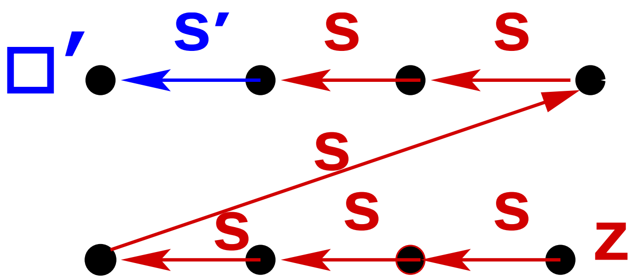

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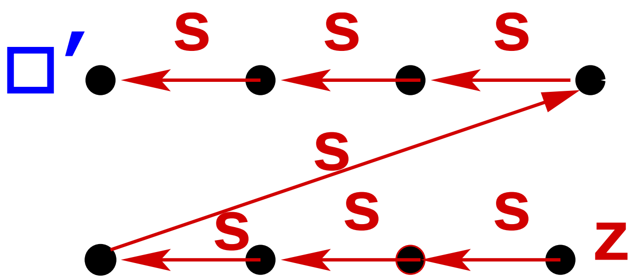

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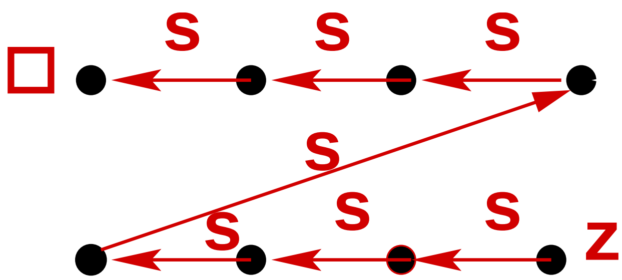

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  - Composition: $P; Q$
  - Branching: $\mathbf{if}\,[\mathbf{t} = \mathbf{q}]\,\{P\}\,\{Q\}$
  - Iteration: $\mathbf{do}\ [T]\{P\} \quad (T \subset V)$
- Semantics: Iteration repeats $\leqslant |\sigma(T)|$ times.

Programs

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  A fnct over $\mathbb{N}$ is computable by a program iff it is PR.
- $\Rightarrow$: Every program over molecules for $\mathbb{N}$ yields a PR function because the semantics of programs is coded in PRA.
- $\Leftarrow$: Every PR is computable by a program because the recurrence schema is implementable by a loop with a counter.

Concrete proof theory

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- Concrete formulas have no reference to infinities.
  Occurrence of $\exists$-ff points to an un-named function!

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    Existence of a linear order, extending s, on the domain of z, s.
- $\boxed{\textit{Inequality}}$ on $\mathbb{N}$ is definable as the existence of a non-surjective embedding.

A concrete theory of FDSs

Here is a theory for FDSs
referring only to concrete formulas.

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  Definitional extension: $f_{\vec{u}!v}$.
- ***Unboundedness*** $\boxed{\exists \vec{u}\ \ f^k \vec{u} = \boldsymbol{\omega}}$

The induction rule

- ***Concrete-Induction-Rule*** For φ concrete

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- When φ is $\exists \vec{g}\, \varphi_0[f, g]$ (φ_0 elementary)
  the second premise is concrete, since it is equivalent to

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- The induction ***axiom*** for concrete φ is ***not*** concrete:
  The premise is of the Induction Schema is

$$\forall f\; \forall \vec{u}, v\; \varphi[f] \to \varphi[f_{\vec{u}!v}]$$

so $\forall$ counts as a positive existential in the schema
but its scope is not concrete.

The crux

FDS is complete for PRA

- Recall: PRA is quantifier-free induction over PR-functions.

- We show that PRA is interpretable in FDS, i.e.:
  1. PR functions are represented by concrete FDS-formulas
  2. Quantifier-free formulas of PRA are interpreted by concrete FDS-formulas.
  3. The interpretation of PRA induction for these formulas is derived from FDS induction.

Representing PR functions

- Every PR function is computable by an FDS-program, and therefore representable by a concrete FDS-formula.
  - Proved by discourse induction on the PR definition.
  - Each step uses formal FDS-induction to prove the existence of a computation trace.

Interpreting quantifier-free numeric formulas

- PRA equality is represented in FDS by ***isomorphism***.
- Isomorphism and numeric-inequality
  are definable by concrete formulas,
  so qf formulas of PRA are interpreted in FDS
  by concrete formulas.

Interpreting PRA induction

- By Parson's Theorem PRA induction is captured by the Σ_1 induction ***rule***.
- Since $\mathbb{N}$ is definable by a concrete formula,
  Σ_1 formulas are interpreted by concrete formulas.

Conversely, FDS is sound for PRA

- Since all ff's are codable in PRA as numbers,
  FDS is interpretable in PRA,
  with concrete fmls interpreted as existential fmls of PRA,
  and so admissible in PRA by Parson's Theorem.

In a word...

We focused on the “hardware” of finitism,

constructed a first-order theory of that hardware,

and showed that it is proof-theoretically equivalent

to primitive recursive arithmetic.

In a word, Tait’s Thesis is vindicated!