An algebraic representation of information types

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The goals of the talk

- to explain the notion of information type
- to show that information types can be combined by logical operators
- to show that there are logical relations among information types
- to develop an algebraic semantics for the logic of information types

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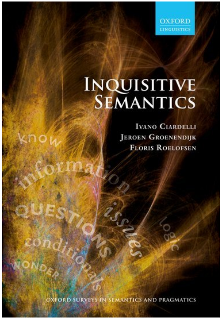
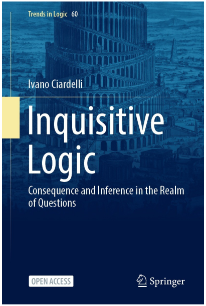
Inquisitive semantics



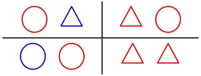


Questions are types of propositions (information types)

Ciardelli, I. (2018) Questions as information types

Information tokens vs. information types

Possible states:



Information tokens:

a is a circle, b is a triangle, a is red, . . .

Information types:

shape of a, shape of b, colour of a, colour of b

- *a is a triangle* $\models_C$ *b is red*
- *a is a circle* $\not\models_C$ *b is red*
- *colour of b, shape of a* $\models_C$ *colour of a*
- *colour of b, shape of a* $\not\models_C$ *shape of b*

Information tokens vs. information types

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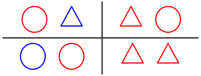

Information tokens:

a is a circle, b is a triangle, a is red, ...

Information types:

shape of a, shape of b, colour of a, colour of b

- *a is a triangle* $\vDash_C$ *b is red*
- *a is a circle* $\nvDash_C$ *b is red*
- *colour of b, shape of a* $\vDash_C$ *colour of a*
- *colour of b, shape of a* $\nvDash_C$ *shape of b*

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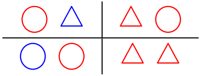

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- *colour of b, shape of a* $\models_C$ *colour of a*
- *colour of b, shape of a* $\not\models_C$ *shape of b*

Combining information types: conjunction

type:

- *the shape of a and the colour of b*

its tokens:

- *a is a circle and b is blue*
- *a is a triangle and b is red*

  ⋮

Combining information types: disjunction

type:

- *the shape of a or the colour of b*

its tokens:

- *a is a circle*
- *b is blue*

  ⋮

Combining information types: implication

type:

- *type of dependencies of the shape of a on the colour of b*

its tokens:

- *if b is red then a is a circle, if b is blue then a is a triangle, if b is green . . .*
- *if b is green then a is a triangle, if b has any other colour then a is a circle*

  ⋮

Combining information types: implication

Tokens of type φ:

- *Kateřina got 0 points, . . ., Kateřina got 100 points*

Tokens of type ψ:

- *Kateřina passed the exam, Kateřina didn't pass the exam*

Tokens of type $\varphi \to \psi$:

- *If Kateřina got* > 60 *points then she passed the exam and if she got* ≤ 60 *points she did not pass*
- *If Kateřina got* > 50 *points then she passed the exam and if she got* ≤ 50 *points she did not pass*

  $\vdots$

Quantifiers: all

type:

- *the colour of all objects*

its tokens:

- *a is red and b is blue and c is . . .*
- *all objects are green*

  ⋮

Quantifiers: some

type:

- *the colour of some objects*

its tokens:

- *a is red*
- *b is blue*

⋮

From types to propositions

Given a type φ we can form the following proposition:

- *some token of the the type* φ *holds* ($\circ\varphi$)

First-order language

The language of types:

$$\varphi, \psi ::= \bot \mid Pt_1 \ldots t_n \mid \varphi \wedge \psi \mid \varphi \rightarrow \psi \mid \forall x \varphi \mid \psi \Vvee \varphi \mid \exists\!\!\exists x \varphi \mid \circ \varphi$$

The sublanguage of tokens:

$$\alpha, \beta ::= \bot \mid Pt_1 \ldots t_n \mid \alpha \wedge \beta \mid \alpha \rightarrow \beta \mid \forall x \alpha \mid \circ \varphi$$

Defined symbols:

$$\neg \varphi =_{def} \varphi \rightarrow \bot, \varphi \leftrightarrow \psi =_{def} (\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$$

$$\varphi \vee \psi =_{def} \circ(\varphi \Vvee \psi), \exists x \varphi =_{def} \circ \exists\!\!\exists x \varphi$$

In inquisitive logic:

- $Pa \mathbin{\vee\!\!\!\vee} Qa$ represents the question *whether a has the property P or the property Q*
- $\circ(Pa \mathbin{\vee\!\!\!\vee} Qa)$ would be the presupposition of the question, i.e. the proposition *that a has the property P or the property Q*
- $\exists\!\!\!\exists x Px$ represents the question *what is an object that has the property P*
- $\circ\exists\!\!\!\exists x Px$ would be the presupposition of the question, i.e. the proposition *that there is an object that has the property P*

The reformulation in terms of types:

- $Pa \mathbin{\vee\!\!\vee} Qa$ represents the type with tokens Pa, Pb,
- $\circ(Pa \mathbin{\vee\!\!\vee} Qa)$ is the statement that some token of the type $Pa \mathbin{\vee\!\!\vee} Qa$ holds: i.e. the proposition $Pa \vee Qa$
- $\exists\!\!\exists x Px$ is the type of instances of Px
- $\circ\exists\!\!\exists x Px$ is the statement that some token of the type $\exists\!\!\exists x Px$ holds: i.e. the proposition $\exists x Px$

Some formalizations

- *the shape of a:* $\exists x Sxa$
- *the shape of a and the colour of b:* $\exists x Sxa \wedge \exists y Cyb$
- *the shape of a or the colour of b:* $\exists x Sxa \vee\!\!\!\vee \exists y Cyb$
- *the colour of all objects:* $\forall x \exists y Cyx$
- *the colour of some objects:* $\exists x \exists y Cyx$
- *dependence of the shape of a on the colour of b:* $\exists y Cyb \rightarrow \exists x Sxa$
- *the colour of a is dependent on the colour of b:* $\circ(\exists y Cyb \rightarrow \exists x Sxa)$

Complete Heyting algebra

A complete Heyting algebra (cHA) is any structure

$$\mathcal{H} = \langle H, \bigvee, \bigwedge, \Rightarrow, 0 \rangle,$$

where

- $\langle H, \bigvee, \bigwedge \rangle$ is a complete lattice,
- 0 is its least element
- $\Rightarrow$ is a relative pseudocomplement, i.e. a binary operation on H satisfying the residuation condition:

$$u \leq s \Rightarrow t \text{ iff } u \wedge s \leq t.$$

Complete Heyting algebras correspond to "frames" of point-free topology

Complete Heyting algebras coincide with complete lattices satisfying the following infinitary distributive law:

$$s \wedge \bigvee_{i \in I} t_i = \bigvee_{i \in I} (s \wedge t_i).$$

In every such lattice, relative pseudocomplement satisfying the residuation condition can be defined as follows:

$$s \Rightarrow t = \bigvee \{u \in H \mid s \wedge u \leq t\}.$$

$\circ$ is a lax modality (Fairtlough 1997)

(a) $\models \varphi \to \circ\varphi$

(b) $\models \circ\circ\varphi \leftrightarrow \circ\varphi$

(c) $\models \circ(\varphi \wedge \psi) \leftrightarrow (\circ\varphi \wedge \circ\psi)$

(d) $\models \circ\bot \leftrightarrow \bot$

Nucleus (the algebraic counterpart of the lax modality)

A nucleus on a Heyting algebra $\mathcal{H}$ is a function $j : H \to H$ such that for each $s, t \in H$:

(a) $s \leq j(s)$,
(b) $j(j(s)) = j(s)$,
(c) $j(s \wedge t) = j(s) \wedge j(t)$.

(every nucleus is a closure operator)

A nucleus is dense if $j(0) = 0$.

Nuclear cHAs

A nuclear cHA (ncHA) is a cHA equipped with a nucleus.

- If $\mathcal{H}$ is an ncHA, the set $jH = \{j(s) \mid s \in H\}$ of all its fixed points will be called the core of $\mathcal{H}$.
- The core elements will be called propositions.

Kripkean ncHAs

Let $\mathcal{H} = \langle H, \bigvee, \bigwedge, \Rightarrow, 0\rangle$ be a cHA. Take the structure

$$Dw(\mathcal{H}) = \langle DwH, \bigcup, \bigcap, \Rrightarrow, \{0\}, dj\rangle$$

where

- DwH is the set of all non-empty downsets of $\mathcal{H}$,
- $\bigcup$ and $\bigcap$ are (infinitary) union and intersection,
- $\Rrightarrow$ is defined as follows:

$$X \Rrightarrow Y = \bigcup\{Z \in DwH \mid Z \cap X \subseteq Y\},$$

- and dj as follows:

$$dj(X) = \downarrow \bigvee X.$$

We will call these structures Kripkean ncHAs.

Propositions

Propositions are closed under implication, conjunction and universal quantification. Moreover, contradiction is a proposition.

Claim

The core of any ncHA is closed under $\bigwedge$ and $\Rightarrow$. In dense ncHAs, the core contains 0.

Propositions form an cHA

If $\mathcal{H} = \langle H, \bigvee, \bigwedge, \Rightarrow, 0, j \rangle$ is a dense ncHA, we can define the structure

$$j\mathcal{H} = \langle jH, \bigvee^j, \bigwedge, \Rightarrow, 0 \rangle$$

where

- $\bigwedge, \Rightarrow, 0$ are taken form $\mathcal{H}$ (restricted to the core),
- and $\bigvee^j X = j(\bigvee X)$, for all $X \subseteq H$.

Claim

$j\mathcal{H}$ is a cHA.

First-order frames

- by a first-order algebraic frame we will understand a pair $\mathcal{F} = \langle \mathcal{H}, D \rangle$, where $\mathcal{H}$ is an ncHA and D is a non-empty set (the domain of quantification).
- a valuation in $\mathcal{F}$ is defined as a function V which assigns to any n-ary predicate P a function $V(P) : D^n \to H$
- we say that a valuation V is informative, if for any n-ary predicate P we have $V(P) : D^n \to jH$

First-order models

- a first-order algebraic model is an algebraic frame equipped with a valuation
- a regular algebraic model is an algebraic model in which the valuation is informative.
- a Kripkean algebraic model is a regular algebraic model based on a Kripkean ncHA.

Evaluation

An evaluation (asignment) in U is a function that assigns to each variable of the language an element of D. If e is an evaluation, x a variable, and $a \in D$, then $e(a/x)$ is the evaluation that assigns a to x and $e(y)$ to any other variable y. For any term t, $V^e(t)$ is identical with $V(t)$ if t is a name, and with $e(t)$ if t is a variable.

Algebraic value of a formula in a ncHA

- $|\bot|_e^{\mathcal{N}} = 0$,
- $|Pt_1 \ldots t_n|_e^{\mathcal{N}} = V(P)(V^e(t_1), \ldots, V^e(t_n))$,
- $|\varphi \wedge \psi|_e^{\mathcal{N}} = |\varphi|_e^{\mathcal{N}} \wedge |\psi|_e^{\mathcal{N}}$,
- $|\varphi \to \psi|_e^{\mathcal{N}} = |\varphi|_e^{\mathcal{N}} \Rightarrow |\psi|_e^{\mathcal{N}}$,
- $|\forall x \varphi|_e^{\mathcal{N}} = \bigwedge_{a \in D} |\varphi|_{e(a/x)}^{\mathcal{N}}$,
- $|\circ\varphi|_e^{\mathcal{N}} = j(|\varphi|_e^{\mathcal{N}})$
- $|\varphi \Vvee \psi|_e^{\mathcal{N}} = |\varphi|_e^{\mathcal{N}} \vee |\psi|_e^{\mathcal{N}}$,
- $|\exists x \varphi|_e^{\mathcal{N}} = \bigvee_{a \in D} |\varphi|_{e(a/x)}^{\mathcal{N}}$.

Validity

- an L-formula φ is e-valid in $\mathcal{N}$, if $|\varphi|_e^{\mathcal{N}} = 1$
- φ is valid in $\mathcal{N}$ if for every evaluation e in $\mathcal{N}$, φ is e-valid in $\mathcal{N}$
- φ is valid in an algebraic frame if it is valid in every algebraic model based on that frame.

Logics

- the logic of all algebraic models is first-order lax logic
- the logic of all regular algebraic models is first-order lax logic plus the following axiom:

$$\circ\alpha \to \alpha, \text{ for elementary formulas}$$

- the logic of all Kripkean models is first-order intuitionistic inquisitive logic validating:

$$(\alpha \to (\psi \mathbin{\vee\!\!\!\vee} \chi)) \to ((\alpha \to \psi) \mathbin{\vee\!\!\!\vee} (\alpha \to \chi))$$

$$(\alpha \to \exists x\psi) \to \exists x(\alpha \to \psi), \text{ if } x \notin FV(\alpha)$$

Representing functions

Let $X, Y \subseteq jH$ and $f : X \to Y$. Let

$$s_f = \bigwedge_{t \in X} (t \Rightarrow f(t))$$

The element s_f represents the function f in $\mathcal{H}$.

Representation of types

$$\mathcal{T}_e(Pt_1 \dots t_n) = \{|Pt_1 \dots t_n|_e\},\ \mathcal{T}_e(\bot) = \{0\},$$

$$\mathcal{T}_e(\varphi \wedge \psi) = \{s \wedge u \mid s \in \mathcal{T}_e(\varphi), u \in \mathcal{T}_e(\psi)\},$$

$$\mathcal{T}_e(\varphi \to \psi) = \{s_f \mid f : \mathcal{T}_e(\varphi) \to \mathcal{T}_e(\psi)\},$$

$$\mathcal{T}_e(\forall x \varphi) = \{\textstyle\bigwedge_{a \in D} f(a) \mid f \in \prod_{a \in D} \mathcal{T}_{e(a/x)}(\varphi)\},$$

$$\mathcal{T}_e(\circ\varphi) = \{j(\textstyle\bigvee \mathcal{T}_e(\varphi))\},$$

$$\mathcal{T}_e(\varphi \vee\!\!\!\vee \psi) = \mathcal{T}_e(\varphi) \cup \mathcal{T}_e(\psi),$$

$$\mathcal{T}_e(\exists\!\!\exists x \varphi) = \textstyle\bigcup\{\mathcal{T}_{e(a/x)}(\varphi) \mid a \in U\}.$$

One can observe that for any formula φ, $\mathcal{T}^{\mathcal{N}}_{e(a/x)}(\varphi)$ is a set of core elements (i.e. declarative propositions) in $\mathcal{N}$

Typical ncHAs

Let $\mathcal{H} = \langle H, \bigvee, \bigwedge, \Rightarrow, 0, j\rangle$ be an ncHA. We say that $\mathcal{H}$ is typical if j is dense and the following two conditions are satisfied for every $s \in H$ and any collection of indexed elements $t_i, u_{ik} \in H$, where $i \in I$ and $k \in K$ for some index sets I, K:

(a) $j(s) \Rightarrow \bigvee_{i\in I} t_i = \bigvee_{i\in I}(j(s) \Rightarrow t_i)$,

(b) $\bigwedge_{i\in I} \bigvee_{k\in K} j(u_{ik}) = \bigvee_{f:I\to K} \bigwedge_{i\in I} j(u_{if(i)})$.

An algebraic frame (model) is called typical if it is based on a typical ncHA.

Arbitrary elements behave as types of the core elements

Theorem

Let $\mathcal{N} = \langle \mathcal{H}, U, V \rangle$ *be a regular algebraic model,* e *an evaluation in* U*, and* φ *an L-formula. If* $\mathcal{N}$ *is typical then*

$$|\varphi|_e^{\mathcal{N}} = \bigvee \mathcal{T}_e^{\mathcal{N}}(\varphi).$$

A connection between typical and Kripkean ncHAs

Definition

Let $\mathcal{H}_1$, $\mathcal{H}_2$ be ncHAs. A homomorphism from $\mathcal{H}_1$ to $\mathcal{H}_2$ is a function $h : H_1 \to H_2$ which preserves the operations $\bigvee, \bigwedge, \Rightarrow, j$ and 0. We say that $\mathcal{H}_2$ is a homomorphic core image of $\mathcal{H}_1$ if $j_2 H_2 = h(j_1 H_1)$.

Theorem

An ncHA is typical iff it is a homomorphic core image of a Kripkean ncHA.

Corollary

The logic of typical algebraic models coincide with the logic of regular Kripkean models (i.e with intuitionistic inquistive logic).

Future work

- deductive characterization of the logic of information types
- algebraic properties of typical algebras
- connections to type theory
- expansion to the substructural setting
- philosophical aspects and applications