Variants of the prenex normal form game

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Ehrenfeucht–Fraïssé game

Fraïssé 1954 [Fra54] characterized the first-order equivalence up to a fixed quantifier rank k of two structures in terms of a system of partial isomorphisms satisfying certain extension conditions. Later, Ehrenfeucht 1961 [Ehr61] presented a game of k rounds that characterizes the same thing. Nowadays, these are regarded just as different formulations of the same matter, and the game is called the *Ehrenfeucht–Fraïssé game*.

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This game has proved to be an invaluable tool in the study of expressive power of the first order logic, over the finite structures even a dominant one. Several variants have been introduced, e.g., Barwise 1977 [Bar77] introduced the logic which is now called *finite variable logic* and introduced an algebraic system that characterizes its equivalence up to k variables. The corresponding game is called the *pebble game*.

Quantifier rank vs size of formulas

However, quantifier rank is an extremely rough measure for size of formulas: Consider the hereditarily finite part or the set-theorical hierarchy: V_n is the set of rank less than n sets, so by recursion, $V_0 = \emptyset$ and $V_{n+1} = \mathcal{P}(V_n)$, for $n \in \mathbb{N}$. Denote by twr the exponential tower function, i.e., $\mathrm{twr}(0) = 0$ and $\mathrm{twr}(n+1) = 2^{\mathrm{twr}(n)}$, for $n \in \mathbb{N}$.

- Every element of V_{n+1} is definable in the structure $(V_{n+1}, \in)$ by a formula of quantifier rank n.
- Since $|V_{n+1}| = \mathrm{twr}(n)$, there are at least $\mathrm{twr}(n)$ non-equivalent formulas of quantifier rank n.

For this reason it is natural to consider finer measures for the size of formulas. The *total number of quantifiers* in a formula is such a measure.

Variants of EF-game

Accordingly, there has been an increasing demand for methods related to finer measures of complexity of formulas than quantifier rank and number of variables (which remain important). An important step towards this end was Immerman 1981 [Imm81] which introduced a version of Ehrenfeuch–Fraïssé game that uses the number of quantifiers instead of quantifier rank as a parameter. This game was forgotten for some 40 years until Fagin-Lenchner-Regan-Vyas 2021 [FLRV21] re-invented it.

Unlike Immerman 1981, Fagin et al 2021 also gave applications for the game: they proved exact bounds for the number of quantifiers needed for separating linear orders of length at most n from those of length greater than n. The study of the Immerman game has been continued by [FLVW22, CFI+23, CFI+24].

Immerman game is designed for counting the number of quantifiers needed to express a property under study. A quite extreme case, in terms of the resources related to a formula, is to calculate the size of the formula. Formula size games were first studied by Hella and Väänänen [HV15], and later by Hella and Vilander [HV16, HV19, HV22]. One common feature of the Immerman game and the formula size game is that, whereas Ehrenfeucht–Fraïssé game is played between a pair $(\mathfrak{A}, \mathfrak{B})$ of structures, these games based on finer resources are played by classes $(\mathcal{A}, \mathcal{B})$ of structures, and this seems to be an intrinsic feature of them.

Introduction to the prenex normal form game

Hella–Luosto 2024 [HL24] defined three other games that are equivalent to the Immerman game in the sense that they characterize definability by sentences with n quantifiers. In this talk I explain how one these games works. The main difference of this game from the Immerman game is that this game is naturally between pair of structures.

Semantic game

As part of the prenex normal form game, we need the well-know semantic game in the special case where the sentence is of the form $\overline{Q}\bar{x}\vartheta(\bar{x})$ with the formula $\vartheta(\bar{x})$ quantifier-free.

Definition

Let $\overline{Q}\bar{x} = Q_0 x_0 \dots Q_{n-1} x_{n-1}$ where $Q_i \in \{\forall, \exists\}$ and let $\vartheta(\bar{x})$ be a quantifier-free formula. $\mathrm{SG}(\mathfrak{A}, \overline{Q}\bar{x}\vartheta(\bar{x}))$ has n rounds, and is played by $\exists$ and $\forall$ as follows:

- Assume that rounds $0, \dots, i-1$ have been played and $i < n$. If $Q_i = \exists$, then $\exists$ chooses an element $a_i \in \mathrm{Dom}(\mathfrak{A})$. Otherwise $\forall$ chooses $a_i \in \mathrm{Dom}(\mathfrak{A})$.

$\exists$ wins the play if $\mathfrak{A} \models \vartheta[\bar{a}]$, where $\bar{a} = (a_0, \dots, a_{n-1})$. Otherwise $\forall$ wins.

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It is well known that the semantic game characterizes the satisfaction relation: $\mathfrak{A} \models \overline{Q}\bar{x}\vartheta(\bar{x})$ if and only if $\exists$ has a winning strategy in $\mathrm{SG}(\mathfrak{A}, \overline{Q}\bar{x}\vartheta(\bar{x}))$. Note that ϑ has an effect only on the winning condition, note on other rules. As part of the larger game, we use thus $\mathrm{SG}(\mathfrak{A}, \overline{Q}\bar{x}\Box)$.

Prenex normal form game

In Hella and Luosto 2024, the following game was proved to be equivalent with the Immerman game.

Definition ([HL24])

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(1) S chooses a quantifier prefix $\overline{Q}\bar{x}$ of length n.

(2) S ($= \exists$) and D ($= \forall$) play $\mathrm{SG}(\mathfrak{A}, \overline{Q}\bar{x}\,\square)$ repeatedly. Let $\bar{a}_i \in A^n$ be the tuple formed by the choices in the i-th repetition. If $\bar{a}_j \mapsto \bar{a}_i$ is a partial isomorphism for some $j < i$, then the players move to step (3); otherwise they play the next repetition of $\mathrm{SG}(\mathfrak{A}, \overline{Q}\bar{x}\,\square)$.

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(3) D $(= \exists)$ and S $(= \forall)$ play $\mathrm{SG}(\mathfrak{B}, \overline{Q}\bar{x}\,\square)$ once. Let $\bar{b} \in B^n$ be the tuple formed by the choices of the players.

(4) D wins the play if $\bar{a}_i \mapsto \bar{b}$ is a partial isomorphism for some i.

Prenex normal form game

The game $\mathrm{PNFG}_{\overline{Q}}(\mathfrak{A}, \mathfrak{B})$ consisting of steps (2)–(4) was introduced by Hella and Luosto 99 (unpublished).

We proved that D has a winning strategy in the game $\mathrm{PNFG}_{\overline{Q}}(\mathfrak{A}, \mathfrak{B})$ if and only if for all quantifier-free ϑ, $\mathfrak{A} \models \overline{Q}\bar{x}\,\vartheta$ implies $\mathfrak{B} \models \overline{Q}\bar{x}\,\vartheta$.

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Theorem (Hella and Luosto 24)

The following conditions are equivalent

(1) *There is a sentence $\varphi \in \mathrm{FO}$ with at most n quantifiers that separates $\mathfrak{A}$ from $\mathfrak{B}$.*

(2) *S has a winning strategy in $\mathrm{PNFG}_n(\mathfrak{A}, \mathfrak{B})$.*

Proof. Assume that there is a sentence φ with at most n quantifiers which separates $\mathfrak{A}$ from $\mathfrak{B}$. Then S has the following winning strategy in the game $\mathrm{PNFG}_n(\mathfrak{A}, \mathfrak{B})$.

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First, we can assume that φ is of the form $\overline{Q}\bar{x}\,\vartheta$ for some $\overline{Q}\bar{x}$ of length n and quantifier free ϑ. S plays the prefix $\overline{Q}\bar{x}$ in step (1) of the game $\mathrm{PNFG}_n(\mathfrak{A}, \mathfrak{B})$.

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Since φ separates structures, $\mathfrak{A} \models \overline{Q}\bar{x}\,\vartheta$ and $\mathfrak{B} \not\models \overline{Q}\bar{x}\,\vartheta$. Thus, $\exists$ has a w.s. in $\mathrm{SG}(\mathfrak{A}, \overline{Q}\bar{x}\,\vartheta)$. S uses this w.s. in all repetitions of $\mathrm{SG}(\mathfrak{A}, \overline{Q}\bar{x}\,\Box)$ in step (2). This guarantees that $\mathfrak{A} \models \vartheta[\bar{a}_i]$ for every tuple $\bar{a}_i$ formed in step (2).

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Similarly $\forall$ has a w.s. in $\mathrm{SG}(\mathfrak{B},\overline{Q}\bar{x}\,\vartheta)$. S uses this w.s. for playing $\mathrm{SG}(\mathfrak{B},\overline{Q}\bar{x}\,\Box)$ in step (3). Thus $\mathfrak{B} \not\models \vartheta[\bar{b}]$ for the resulting tuple $\bar{b} \in B^n$.

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Clearly none of the mappings $\bar{a}_i \mapsto \bar{b}$ is a partial isomorphism $\mathfrak{A} \to \mathfrak{B}$.

Proof (continued). Assume then that there is no sentence φ with at most n quantifiers which separates $\mathfrak{A}$ from $\mathfrak{B}$. We show that then D has a winning strategy in $\mathrm{PNFG}_n(\mathfrak{A}, \mathfrak{B})$.

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Let $\overline{Q}\bar{x}$ be the prefix chosen by S in step (1). In step (2), D plays as follows:

- She uses an arbitrary strategy in the first repetition of $\mathrm{SG}(\mathfrak{A}, \overline{Q}\bar{x}\,\square)$.
- Assume that k repetitions of $\mathrm{SG}(\mathfrak{A}, \overline{Q}\bar{x}\,\square)$ have been played. Let

$$\vartheta_k(\bar{x}) := \bigvee_{1 \le i \le k} \psi_{\mathfrak{A},\bar{a}_i}(\bar{x}).$$

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$$\vartheta_k(\bar{x}) := \bigvee_{1 \le i \le k} \psi_{\mathfrak{A}, \bar{a}_i}(\bar{x}).$$

  If $\mathfrak{A} \not\models \overline{Q}\bar{x}\,\vartheta_k$, then $\forall$ has a w.s. in $\mathrm{SG}(\mathfrak{A}, \overline{Q}\bar{x}\,\vartheta_k)$, and D uses this w.s. for playing the $k+1$-th repetition. This guarantees that $\mathfrak{A} \not\models \vartheta_k[\bar{a}_{k+1}]$. Otherwise D plays with an arbitrary strategy.

Note that the players move to step (3) at latest after M_n repetitions, where M_n is the number of different formulas of the form $\psi_{\mathfrak{A},\bar{a}}$.

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- Assume then that the players have moved to step (3) after r repetitions of $\mathrm{SG}(\mathfrak{A}, \overline{Q}\bar{x}\,\Box)$. Then $\mathfrak{A} \models \overline{Q}\bar{x}\,\vartheta_r$, and by assumption $\mathfrak{B} \models \overline{Q}\bar{x}\,\vartheta_r$.

  This means that $\exists$ has a w.s. in $\mathrm{SG}(\mathfrak{B}, \overline{Q}\bar{x}\,\vartheta_r)$. Let D use this w.s. in $\mathrm{SG}(\mathfrak{B}, \overline{Q}\bar{x}\,\Box)$.

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  This means that $\exists$ has a w.s. in $\mathrm{SG}(\mathfrak{B}, \overline{Q}\bar{x}\,\vartheta_r)$. Let D use this w.s. in $\mathrm{SG}(\mathfrak{B}, \overline{Q}\bar{x}\,\Box)$.

  Then $\mathfrak{B} \models \vartheta_r[\bar{b}]$ for the resulting tuple $\bar{b} \in B^n$. Consequently $\mathfrak{B} \models \psi_{\mathfrak{A},\bar{a}_i}[\bar{b}]$ holds for some $1 \leq i \leq r$. But this means that $\bar{a}_i \mapsto \bar{b}$ is a partial isomorphism $\mathfrak{A} \to \mathfrak{B}$, and hence D wins. $\Box$

The type prenex normal form game

Observe that the number of repetitions in step (2) of $\mathrm{PNFG}_n(\mathfrak{A}, \mathfrak{B})$ is tightly related to the number r of disjuncts in the formula $\vartheta_r(\bar{x}) = \bigvee_{1 \leq i \leq r} \psi_{\mathfrak{A}, \bar{a}_i}(\bar{x})$.

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Thus, requiring that S has to force the move to step (3) after r-th repetition (or earlier), we obtain a version $\mathrm{PNFG}^r_n(\mathfrak{A}, \mathfrak{B})$ of the game s.t. TFAE:

(1) φ separates $\mathfrak{A}$ from $\mathfrak{B}$ with $\varphi = \overline{Q}\bar{x}\,\vartheta$, where ϑ is a disjunction of at most r complete atomic types.

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Thus, requiring that S has to force the move to step (3) after r-th repetition (or earlier), we obtain a version $\mathrm{PNFG}_n^r(\mathfrak{A}, \mathfrak{B})$ of the game s.t. TFAE:

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(2) S has a winning strategy in $\mathrm{PNFG}_n^r(\mathfrak{A}, \mathfrak{B})$.

However, *the type prenex normal form game* is not well-behaving, e.g., consider adding dummy variables and quantifiers to the formula. This lead us to the next variant of PNFG.

The disjunctive prenex normal form game

The *disjunctive prenex normal form game* $\mathrm{DPNFG}_{\overline{Q},r}(\mathfrak{A}, \mathfrak{B})$ is like $\mathrm{PNFG}_{\overline{Q}}(\mathfrak{A}, \mathfrak{B})$, except that S chooses a colour $\chi_i \in [r]$ for each tuple $\bar{a}_i$ played in step (2).

The idea is that S uses the colours χ_i to classify the tuples $\bar{a}_i$ according to the disjuncts he uses in the formula $\overline{Q}\bar{x}\,\vartheta$ that is supposed to separate $\mathfrak{A}$ and $\mathfrak{B}$.

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The idea is that S uses the colours χ_i to classify the tuples $\bar{a}_i$ according to the disjuncts he uses in the formula $\overline{Q}\bar{x}\,\vartheta$ that is supposed to separate $\mathfrak{A}$ and $\mathfrak{B}$.

The goal of D is then to make sure that the tuple $\bar{b}$ played in step (3) satisfies one of the disjuncts. The players need some book-keeping mechanism to follow what happens. If α and β are quantifier-free, write

$$\alpha + \beta = \bigwedge_{\gamma \text{ literal},\, \alpha\cup\beta\models\gamma} \gamma.$$

Definition

The *disjunctive prenex normal form game* $\mathrm{DPNFG}_{\overline{Q},r}(\mathfrak{A},\mathfrak{B})$ is played as follows: Before the game we set $\beta_{k,0} = \perp$, for each colour $k \in \{0,\dots r-1\}$.

1) Assume that repetitions $j \in i = \{0,\dots,i-1\}$ of $\mathrm{SG}(\mathfrak{A},\overline{Q}\bar{x}\,\Box)$ have been played, and $\bar{a}_j$ and χ_i, for $j \in i$, have been defined.

   Let $\bar{a}_i$ be the tuple formed in the i-st repetition, and suppose α_i is its complete quantifier-free type. S now chooses the colour $c = \chi_i$. The book-keeping formulas are now updated so that $\beta_{c,i+1} = \beta_{c,i} + \alpha_i$ and $\beta_{k,i+1} = \beta_{k,i}$, otherwise. If there is now change, i.e., $\beta_{c,i+1} = \beta_{c,i}$, the players move to step 2.

Definition

The *disjunctive prenex normal form game* $\mathrm{DPNFG}_{\overline{Q},r}(\mathfrak{A},\mathfrak{B})$ is played as follows: Before the game we set $\beta_{k,0} = \bot$, for each colour $k \in \{0, \ldots r-1\}$.

1) Assume that repetitions $j \in i = \{0, \ldots, i-1\}$ of $\mathrm{SG}(\mathfrak{A}, \overline{Q}\bar{x}\,\Box)$ have been played, and $\bar{a}_j$ and χ_i, for $j \in i$, have been defined.

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2) D and S play $\mathrm{SG}(\mathfrak{B}, \overline{Q}\bar{x}\,\Box)$ once forming a tuple $\bar{b} \in B^n$.

(3) Put $\beta_k = \beta_{k,i}$ where i is the last round played in step 2, for each colour $k \in r$. Then D wins the play if $\mathfrak{B} \models \beta_k$ for some $k \in r$.

A variation of the prenex normal form game

Let $\mathrm{DPNFO}_{\overline{Q},r}$ be the fragment of FO consisting of all formulas of the form $\overline{Q}\bar{x} \bigvee_{\ell\in[r]} \psi_\ell$, each ψ_ℓ is a conjunction of literals.

Theorem

The following conditions are equivalent

1) *There is a sentence* $\varphi \in \mathrm{DPNFO}_{\overline{Q},r}$ *such that* $\mathfrak{A} \models \varphi$*, but* $\mathfrak{B} \not\models \varphi$*,*
2) S *has a winning strategy in* $\mathrm{DPNFG}_{\overline{Q},r}(\mathfrak{A}, \mathfrak{B})$*.*

Takk!

Takk!

Takk!

Tack!

Kiitos!

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