Epsilon Modal Logics

Elio La Rosa (MCMP, LMU)

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In this talk, I introduce Epsilon Modal logics, a new class of modal logics structurally analogous to Hilbert's Epsilon Calculus:

- In Epsilon Calculus, epsilon terms pick a witness for their bound formula, if any;
- In Epsilon Modal logics, epsilon modalities pick a related world satisfying their formula index, if any.

EMLs generalize the choice-functional reasoning of Epsilon Calculus at an intensional level.

I will present some features of EMLs, such as mutual embeddability with Epsilon Calculus, together with an application to connexive conditionals.

Semantics

Hilbert and Bernays (1939) developed Epsilon Calculus in the context of the foundational project known as Hilbert's Program (Zach, 2020).

Epsilon Calculus $\boldsymbol{\varepsilon}\mathbf{P}$ extends a language for (quantifier-free) predicate logic $\mathcal{L}_{\mathbf{P}}$ to a language $\mathcal{L}_{\boldsymbol{\varepsilon}\mathbf{P}}$ with 'epsilon terms' $\varepsilon x\, A$ and $\tau x\, A$.

Epsilon terms are interpreted over total choice functions:

Definition

For any f.o. model $\mathcal{M} = \langle \mathcal{D}, I \rangle$, the choice function $\phi \colon \wp(\mathcal{D}) \to \mathcal{D}$ is s.t. for any $X \subseteq \mathcal{D}$:

$$\phi(X) := \begin{cases} d \in X & \textit{if } X \neq \emptyset \\ d \in \mathcal{D} & \textit{otherwise} \end{cases}$$

- $\varepsilon x\, A$ denotes an object satisfying A (if any):

$$I_{\sigma,\phi}(\varepsilon x\, A) := \phi(\{d \mid \mathcal{M}, \sigma\tfrac{d}{x}, \phi \Vdash A\})$$

- $\tau x\, A$ denotes an object satisfying A (if any) when all objects do:

$$I_{\sigma,\phi}(\tau x\, A) := \phi(\{d \mid \mathcal{M}, \sigma\tfrac{d}{x}, \phi \nVdash A\})$$

Notice that ε- and τ-terms always denote, and that for any $\mathcal{M}$, σ, ϕ:

$$I_{\sigma,\phi}(\varepsilon x\, \neg A) = I_{\sigma,\phi}(\tau x\, A) \quad \text{and} \quad I_{\sigma,\phi}(\varepsilon x\, A) = I_{\sigma,\phi}(\tau x\, \neg A)$$

Truth in a model and validity for $\boldsymbol{\varepsilon}$**P** are defined as follows resp.:

$$\mathcal{M} \vDash A \quad \textit{iff} \quad \forall\sigma\forall\phi\colon \mathcal{M}, \sigma, \phi \Vdash A$$

$$\Gamma \vDash_{\boldsymbol{\varepsilon}\mathbf{P}} C \quad \textit{iff} \quad \forall\mathcal{M}\forall\sigma\forall\phi\colon \forall A \in \Gamma\colon \mathcal{M}, \sigma, \phi \Vdash A \Rightarrow \mathcal{M}, \sigma, \phi \Vdash C$$

Referents of ε- and τ-terms remain indeterminate in evaluations:

Example

Let $\mathcal{M}$ be an $\boldsymbol{\varepsilon}$**P** model s.t. $\mathcal{D} = \{d_1, d_2, d_3\}$ and $I(P) = I(Q) = \{d_1, d_2\}$.

Then, $\mathcal{M} \vDash Q\varepsilon x\, Px$, and:

- $\phi_1(\{d \mid \mathcal{M}, \sigma\frac{d}{x}, \phi_1 \Vdash Px\}) = d_1$ for some ϕ_1;
- $\phi_2(\{d \mid \mathcal{M}, \sigma\frac{d}{x}, \phi_2 \Vdash Px\}) = d_2$ for some ϕ_2;

The fact that witnesses represented by an epsilon term are arbitrary chose allows for:

- embedding f.o. quantification (and more) in Epsilon Calculus;
- expressing indefinite descriptions of objects satisfying a certain property by epsilon terms.

Axiomatization

A calculus for $\varepsilon\mathbf{P}$ is obtained adding the following axioms to (quantifier-free) Predicate logic:

Crit $A(t) \to A(\varepsilon x\, A(x))$

Def $C(\varepsilon x\, A) \leftrightarrow C(\tau x\, \neg A)$

Ext $(A \leftrightarrow B)(\tau x\, (A \leftrightarrow B)/x) \to (C(\varepsilon x\, A) \leftrightarrow C(\varepsilon x\, B))$

No rule of generalization/eigenvariable conditions needed!

Theorem (Soundness and Completeness of $\varepsilon\mathbf{P}$)

$\Gamma \vDash_{\varepsilon\mathbf{P}} A$ *iff* $\Gamma \vdash_{\varepsilon\mathbf{P}} A$

Conservativity

Epsilon Calculus is conservative over (non-quantified) $\mathcal{L}_{\mathbf{P}}$ formulas:

Theorem (1st Epsilon Theorem)

If $\vdash_{\varepsilon\mathbf{P}} A$ and A quantifier- and epsilon-free, then $\vdash_{\mathbf{P}} A$.

Theorem (2nd Epsilon Theorem)

If $\vdash_{\varepsilon\mathbf{P}} A$ and A epsilon-free, then $\vdash_{\mathbf{P}} A$.

A formula is epsilon-free iff no ε- or τ-terms occur in it.

These results do not hold over weaker logic bases, e.g., Intuitionistic Predicate logic.

Expressivity

Epsilon Calculus is strictly more expressive than Predicate logic:

- Quantifiers $\exists$ and $\forall$ are definable over $\mathcal{L}_{\varepsilon \mathbf{P}}$:

$$\exists x\, A :\leftrightarrow A(\varepsilon x\, A/x) \quad \text{and} \quad \forall x\, A :\leftrightarrow A(\tau x\, A/x)$$

- All definable Skolem functions are representable in $\mathcal{L}_{\varepsilon \mathbf{P}}$:

$$\begin{aligned} \forall x \exists y\, A(x,y) \quad &\mapsto_{\mathsf{sk}} \quad \forall x\, A(x, f(x)), \text{ for } f \text{ fresh} \\ &\mapsto_{\varepsilon} \quad \forall x\, A(x, \varepsilon y\, A(x,y)) \end{aligned}$$

I refer to these conservativity, expressivity and arbitrary witness choice properties as allowing for choice-functional reasoning.

Epsilon Modal logics are a new class of modal logics[1] accounting for choice-functional reasoning at an intensional level.

Any EML $\varepsilon\mathbf{M}$ extends a (modal) propositional language $\mathcal{L}_{\mathbf{M}}$ to a language $\mathcal{L}_{\varepsilon\mathbf{M}}$ by 'epsilon modalities' $\langle A \rangle$ and $[A]$.

Epsilon modalities are interpreted over total choice functions again:

Definition

For any Kripke model $\mathcal{M}$ based on a frame $\mathcal{F} = \langle \mathcal{W}, \mathcal{R} \rangle$, the choice functions ϕ: $\wp(\mathcal{W} \to \mathcal{W})$ is s.t. for any $X \subseteq \mathcal{W}$:

$$\phi(X) := \begin{cases} w \in X & \textit{if } X \neq \emptyset \\ w \in \mathcal{W} & \textit{otherwise} \end{cases}$$

[1] AFAIK, only Fitting (1972) sketches (4 pages) a similar approach based on Epsilon Calculus. Chan (1987) develops a comparable axiomatization. Both rely on different languages and semantics with different expressive power.

- w satisfies $\langle A\rangle B$ iff a related world satisfying A (if any) satisfies B:

$$\mathcal{M}, w, \phi \Vdash \langle A\rangle B \quad \textit{iff} \quad w\mathcal{R}w^{\langle A\rangle} \textit{ and } \mathcal{M}, w^{\langle A\rangle}, \phi \Vdash B,$$
$$\textit{for} \;\; w^{\langle A\rangle} := \phi(\{w' \mid w\mathcal{R}w' \textit{ and } \mathcal{M}, w', \phi \Vdash A\})$$

- w satisfies $[A]B$ iff, if a related world (if any) satisfies A when all related worlds do, then it satifies B:

$$\mathcal{M}, w, \phi \Vdash [A]B \quad \textit{iff} \quad w\mathcal{R}w^{[A]} \Rightarrow \mathcal{M}, w^{[A]}, \phi \Vdash B,$$
$$\textit{for} \;\; w^{[A]} := \phi(\{w' \mid w\mathcal{R}w' \textit{ and } \mathcal{M}, w', \phi \nVdash A\})$$

- w satisfies $\langle A\rangle B$ iff a related world satisfying A (if any) satisfies B:

$$\mathcal{M}, w, \phi \Vdash \langle A\rangle B \quad \textit{iff} \quad w\mathcal{R}w^{\langle A\rangle} \textit{ and } \mathcal{M}, w^{\langle A\rangle}, \phi \Vdash B,$$

$$\textit{for} \ \ w^{\langle A\rangle} := \phi(\{w' \mid w\mathcal{R}w' \textit{ and } \mathcal{M}, w', \phi \Vdash A\})$$

$$\textit{i.e.,} \ \ w^{\langle A\rangle} := \varepsilon w' \, (w\mathcal{R}w' \textit{ and } \mathcal{M}, w' \Vdash A)$$

- w satisfies $[A]B$ iff, if a related world (if any) satisfies A when all related worlds do, then it satifies B:

$$\mathcal{M}, w, \phi \Vdash [A]B \quad \textit{iff} \quad w\mathcal{R}w^{[A]} \Rightarrow \mathcal{M}, w^{[A]}, \phi \Vdash B,$$

$$\textit{for} \ \ w^{[A]} := \phi(\{w' \mid w\mathcal{R}w' \textit{ and } \mathcal{M}, w', \phi \nVdash A\})$$

$$\textit{i.e.,} \ \ w^{[A]} := \tau w' \, (w\mathcal{R}w' \Rightarrow \mathcal{M}, w' \Vdash A)$$

Cf. Leitgeb (2023) for applications of Epsilon Calculus to the metatheory.

Notice that ϕ always picks a world, and for any model $\mathcal{M}$, w and ϕ:

$$\mathcal{M}, w, \phi \Vdash \langle A \rangle B \quad \textit{iff} \quad \mathcal{M}, w, \phi \Vdash \neg[\neg A]\neg B$$
$$\mathcal{M}, w, \phi \Vdash [A] B \quad \textit{iff} \quad \mathcal{M}, w, \phi \Vdash \neg\langle\neg A\rangle\neg B$$

Truth at a world and validity for any EML $\varepsilon\mathbf{M}$ are defined as follows resp.:

$$\mathcal{M}, w \vDash A \quad \textit{iff} \quad \forall \phi\colon \mathcal{M}, w, \phi \Vdash A$$
$$\Gamma \vDash_{\varepsilon\mathbf{M}} C \quad \textit{iff} \quad \forall \mathcal{M} \forall w \forall \phi\colon \forall A \in \Gamma\colon \mathcal{M}, w, \phi \Vdash A \Rightarrow \mathcal{M}, w, \phi \Vdash C$$

Since models of any EML $\varepsilon\mathbf{M}$ are obtained defining choice functions over Kripke models, I will refer to $\mathbf{M}$ as the base Modal logic of $\varepsilon\mathbf{M}$.

World referents of epsilon modalities remain indeterminate in evaluations:

Example

Let $\mathcal{M}$ be an EML model s.t. $\mathcal{W} = \{w, w_1, w_2\}$, $\mathcal{R} = wRw_1, wRw_2$, $P, Q \in w_1$ and $P, Q \in w_2$.

Then, $\mathcal{M}, w \vDash \langle P \rangle Q$, and:

- $\phi_1(\{w\mathcal{R}w' \mid \mathcal{M}, w', \phi_1 \Vdash P\}) = w_1$ for some ϕ_1;
- $\phi_2(\{w\mathcal{R}w' \mid \mathcal{M}, w', \phi_2 \Vdash P\}) = w_2$ for some ϕ_2;

The fact that the world witnesses represented by an epsilon modality are arbitrary chosen allows for:

- embedding standard modalities (and more) in EMLs;
- expressing indefinite descriptions of worlds satisfying a certain property by epsilon modalities.

Axiomatization

A calculus for the EML $\varepsilon\mathbf{K}$ is obtained by adding the following over (**K**) Propositional logic :

wCrit	$\langle B\rangle A \to \langle A\rangle A$
Def	$\langle A\rangle C \leftrightarrow \neg[\neg A]\neg C$
$\circ$Dist	$[A](B \circ C) \leftrightarrow ([A]B \circ [A]C),$ *for* $\circ \in \{\wedge, \vee, \to\}$
$\neg$Dist	$\neg[A]B \to [A]\neg B$
Ext	$[A \leftrightarrow B](A \leftrightarrow B) \to ([A]C \leftrightarrow [B]C)$
NEC	If $\vdash A$, then $\vdash [A]A$

Theorem (Soundness and Completeness of $\varepsilon\mathbf{K}$)

$\Gamma \vDash_{\varepsilon\mathbf{K}} A$ *iff* $\Gamma \vdash_{\varepsilon\mathbf{K}} A$

Standard modalities $\Diamond$ and $\Box$ can be defined as follows:

$$\Diamond A :\leftrightarrow \langle A \rangle A \quad \textit{and} \quad \Box A :\leftrightarrow [A]A$$

Example (Axiom K)

$\vdash_{\varepsilon\mathbf{K}} [B \to C](B \to C) \to ([B]B \to [C]C)$

EML extensions of well-known **K** extensions are obtained adding their (translated) characteristic axioms:

D	$[A]A \to \langle A \rangle A$	4	$[A]A \to [[A]A][A]A$
T	$[A]A \to A$	5	$\langle A \rangle A \to [\langle A \rangle A]\langle A \rangle A$

Any resulting $\varepsilon\mathbf{M}$ is sound and complete over its respective class of frames, and conservative over $\mathcal{L}_{\mathbf{M}}$ (proof later).

Some frame conditions based over total choice functions are representable as well, such as Choice Seriality:

$$\forall w : w\mathcal{R}\phi(X), \text{ \textit{for} } X \subseteq \mathcal{W}$$

Choice Seriality implies Seriality, and makes epsilon modalities functional:[2]

F $[A]\neg B \leftrightarrow \neg[A]B$

Axiom F is valid over universal frames (any two worlds relate), and derives in $\varepsilon\mathbf{K}$ a simplified version of Def:

$$\vdash_{\varepsilon\mathbf{M}} \langle A\rangle B \leftrightarrow [\neg A]B \qquad \vdash_{\varepsilon\mathbf{M}} [A]B \leftrightarrow \langle\neg A\rangle B$$

[2] All the logics of Fitting (1972) include axiom F, but no embedding of $\Diamond$ and $\Box$ is claimed there.

Standard Translation

Any formula of a Modal logic **M** over **P**-definable frame conditions is embeddedable in **P** plus axioms expressing such conditions by the 'standard translation' ST indexed by variables:

$$
\begin{aligned}
\mathtt{ST}_x(P) &:= Px \\
\mathtt{ST}_x(\neg A) &:= \neg \mathtt{ST}_x(A) \\
\mathtt{ST}_x(A \circ B) &:= \mathtt{ST}_x(A) \circ \mathtt{ST}_x(B), \quad \textit{for } \circ \in \{\wedge, \vee, \to\} \\
\mathtt{ST}_x(\Diamond A) &:= \exists y\, (xRy \wedge \mathtt{ST}_y(A)) \\
\mathtt{ST}_x(\Box A) &:= \forall y\, (xRy \to \mathtt{ST}_y(A))
\end{aligned}
$$

Theorem (Embedding of K in P)

$\vDash_{\mathbf{K}} A \quad$ *iff* $\quad \vDash_{\mathbf{P}} \forall x\, \mathtt{ST}_x(A)$

The embedding can be extended to formulas of $\varepsilon\mathbf{M}$ over $\varepsilon\mathbf{P}$ plus axioms expressing frame conditions.

The new translation $\varepsilon\mathtt{ST}$ is indexed by ε- and τ-terms as well:

$$
\begin{aligned}
\varepsilon\mathtt{ST}_t(P) &:= Pt \\
\mathtt{ST}_x(\neg A) &:= \neg\mathtt{ST}_x(A) \\
\mathtt{ST}_x(A \circ B) &:= \mathtt{ST}_x(A) \circ \mathtt{ST}_x(B), \quad \textit{for } \circ \in \{\wedge, \vee, \rightarrow\} \\
\varepsilon\mathtt{ST}_t(\langle A\rangle B) &:= tRt^{\langle A\rangle} \wedge \varepsilon\mathtt{ST}_{t^{\langle A\rangle}}(B), \quad \textit{for } t^{\langle A\rangle} := \varepsilon x\,(tRx \wedge \varepsilon\mathtt{ST}_x A) \\
\varepsilon\mathtt{ST}_t([A]B) &:= tRt^{[A]} \rightarrow \varepsilon\mathtt{ST}_{t^{[A]}}(B), \quad \textit{for } t^{[A]} := \tau x\,(tRx \rightarrow \varepsilon\mathtt{ST}_x A)
\end{aligned}
$$

Theorem (Embedding of $\varepsilon\mathbf{K}$ in $\varepsilon\mathbf{P}$)

$\vDash_{\varepsilon\mathbf{K}} A$ *iff* $\vDash_{\varepsilon\mathbf{P}} \varepsilon\mathtt{ST}_{\tau x\, \varepsilon\mathtt{ST}_x(A)}(A)$

By definition of $\Diamond$ and $\Box$ in EMLs and definition of $\exists$ and $\forall$ in ε**P**, the clauses of $\mathtt{ST}$ can be recovered:

$$\varepsilon\mathtt{ST}_t(\langle A\rangle A) \ := \ t\mathcal{R}t^{\langle A\rangle} \wedge \varepsilon\mathtt{ST}_{t^{\langle A\rangle}}(A), \ \textit{for } t^{\langle A\rangle} := \varepsilon x\,(t\mathcal{R}x \wedge \varepsilon\mathtt{ST}_x A)$$

i.e., $\varepsilon\mathtt{ST}_t(\Diamond A) := \ \exists x\,(t\mathcal{R}x \wedge \varepsilon\mathtt{ST}_x(A))$

$$\varepsilon\mathtt{ST}_t([A]A) \ := \ t\mathcal{R}t^{[A]} \rightarrow \varepsilon\mathtt{ST}_{t^{[A]}}(B), \ \textit{for } t^{[A]} := \tau x\,(t\mathcal{R}x \rightarrow \varepsilon\mathtt{ST}_x A)$$

i.e., $\varepsilon\mathtt{ST}_t(\Box A) := \ \forall x\,(t\mathcal{R}x \rightarrow \varepsilon\mathtt{ST}_x(A))$

Therefore, the conservativity of ε**P** over **P** spreads over the modal case:

Theorem

Any ε**M** over **P**-definable frame conditions is a conservative extension of its base Modal logic **M**.

As a corollary, EMLs are strictly more expressive than their Modal base.

Modal Translation

Fitting (2002) provides a converse embedding $\mathtt{MT}$ of **P** over **λS5**, i.e., quantifier-free predicate **S5** over a language containing λ predicate abstraction and intensional variables, denoted by i:

$$\begin{aligned}
\mathtt{MT}(P) &:= \Box P \\
\mathtt{MT}(\neg A) &:= \Box\neg\mathtt{MT}(A) \\
\mathtt{MT}(A \circ B) &:= \Box(\mathtt{MT}(A) \circ \mathtt{MT}(B)), \quad \textit{for } \circ \in \{\wedge, \vee, \rightarrow\} \\
\mathtt{MT}(\exists x\, A) &:= \Diamond(\lambda x\, \mathtt{MT}(A))i, \quad \textit{for } i \textit{ fresh} \\
\mathtt{MT}(\forall x\, A) &:= \Box(\lambda x\, \mathtt{MT}(A))i, \quad \textit{for } i \textit{ fresh}
\end{aligned}$$

The embedding relies on that of Predicate logic over **S5**.

Theorem (Embedding of P in λS5)

$\vDash_{\mathbf{P}} A \quad \textit{iff} \quad \vDash_{\boldsymbol{\lambda}\mathbf{S5}} \mathtt{MT}(A)$

Fitting's embedding can be simplified and extended to $\varepsilon\mathbf{P}$ over $\varepsilon\boldsymbol{\lambda}\mathbf{U}$, i.e., quantifier-free Predicate Modal logic of Universal frames over a language containing λ and intensional variables:

$$
\begin{aligned}
\varepsilon\mathtt{MT}(P) &:= [P]P, \quad \textit{for } P \textit{ epsilon-free} \\
\varepsilon\mathtt{MT}(P(\varepsilon x\, A/x)) &:= \langle A(i/x)\rangle(\lambda x\, \varepsilon\mathtt{MT}(P))i, \quad \textit{for } A \textit{ epsilon-free, } i \textit{ fresh} \\
\varepsilon\mathtt{MT}(P(\tau x\, A/x)) &:= [A(i/x)](\lambda x\, \varepsilon\mathtt{MT}(P))i, \quad \textit{for } A \textit{ epsilon-free, } i \textit{ fresh} \\
\varepsilon\mathtt{MT}(\neg A) &:= \neg\varepsilon\mathtt{MT}(A) \\
\varepsilon\mathtt{MT}(A \circ B) &:= \varepsilon\mathtt{MT}(A) \circ \varepsilon\mathtt{MT}(B), \quad \textit{for } \circ \in \{\wedge, \vee, \rightarrow\}
\end{aligned}
$$

Theorem (Embedding of $\varepsilon\mathbf{P}$ in $\varepsilon\boldsymbol{\lambda}\mathbf{U}$)

$\vDash_{\varepsilon\mathbf{P}} A \quad \textit{iff} \quad \vDash_{\varepsilon\boldsymbol{\lambda}\mathbf{U}} \varepsilon\mathtt{MT}(A)$

The previous embeddings show how Epsilon Calculus applications can be reinterpreted in Epsilon Modal logics:

- by $\varepsilon\mathtt{ST}$, indefinite descriptions of worlds are expressible as indefinite descriptions of objects;
- by $\varepsilon\mathtt{MT}$, indefinite descriptions of objects are expressible as interpretations at indefinitely described worlds.

This shows that not only choice-functional reasoning can be represented in EMLs, but that it generalizes at the intensional level.

As an example, consider choice-functional reasoning allowing the representation of conceptual idealization processes:

- epsilon terms as Hilbert's ideal elements of mathematical properties in his foundational program;
- epsilon terms as Carnap's (1961) explicit definitions of theoretical terms in his reconstruction of scientific theories.

These accounts can be generalized as well, allowing for defining context of evaluation satisfying certain theoretical properties:

Example

Monoids structures $\mathtt{M}$ are axiomatized as follows:

- $\forall x \forall y \forall z\, (x \circ y) \circ z = x \circ (y \circ z)$
- $\forall x\, x \circ \mathsf{e} = \mathsf{e} \circ x = x$

Denote the conjunction of the above as $\mathtt{Axs}(\mathtt{M})$. Commutative monoids add:

Comm $\forall x \forall y\, x \circ y = y \circ x$

EMLs also allow commutative monoids structures to be accounted for as monoids interpreted in a context witnessing Comm:

$$\langle \forall x \forall y\, x \circ y = y \circ x \rangle \mathtt{Axs}(\mathtt{M})$$

The above is not equivalent to $\mathtt{Axs}(\mathtt{M})$ plus Comm.

Epsilon modalities consist of connectives indexed by formulas. Chellas (1975) interpreted this kind of modalities as antecedent of conditionals.

In turn, these represent relations over their intensions in Kripke models:

$$\mathcal{M}, w \Vdash A > B \quad \textit{iff} \quad \forall w' \colon w \mathcal{R}_{|A|} w' \Rightarrow \mathcal{M}, w' \Vdash B,$$
$$\textit{for} \;\; |A| = \{w \in \mathcal{W} \mid \mathcal{M}, w \Vdash A\}$$

Chellas' Normal Conditional logic **CK** coincides with poly-modal **K** plus a rule of extensionality:

K $(A > (B \to C)) \to ((A > B) \to (A > C))$

NEC If $\vdash A$, then $\vdash B > A$

EA If $\vdash A \leftrightarrow B$, then $\vdash (A > C) \leftrightarrow (B > C)$

Epsilon modalities of ε**K**F can be represented in a logic **C**ε**K**F over a class of Chellas' frames an embedding CT as follows:

$$\langle A \rangle B := \neg(A > \neg B) \qquad [A]B := \neg A > B$$

The frame condition (from Unterhuber and Schurz, 2014) validating the embedding interprets $\mathcal{R}$ as an arbitrary choice function ϕ by the following:

(d) $\forall w \exists w' \colon w \mathcal{R}_X w'$

(cem) $\forall w \forall w' \forall w'' \colon (w \mathcal{R}_X w' \text{ and } w \mathcal{R}_X w'') \Rightarrow w' = w''$

(mod) $\forall w \colon \forall w' \colon (w \mathcal{R}_{(\mathcal{W} \setminus X)} w' \Rightarrow w' \in X) \Rightarrow \forall w'' \colon (w \mathcal{R}_Y w' \Rightarrow w' \in X)$

Theorem (Embedding of εKF in CεKF)

$\vDash_{\varepsilon \mathbf{K}\mathsf{F}} A$ *iff* $\vDash_{\mathbf{C}\varepsilon\mathbf{K}\mathsf{F}} \mathtt{CT}(A)$

Connexivity

Connexive conditionals (Wansing, 2023) trace back to ideas of Chrysippus, Aristotle and Boethius among others, and aim at preserving a notion of content-connection between antecedent and consequent.

Some connexive principles are contra-classical. In their strong hyperconnexive (Sylvan, 1989) version, they amount to:

AT	$\neg(\neg A > A)$
BE	$(A > B) \gtrless \neg(A > \neg B)$
NonSym	$\not\models (A > B) \to (B > A)$
NonConjSimp	$\not\models (A \wedge B) > A$

These principles are notoriously difficult to characterize (especially over a Classical base) and to interpret intuitively (especially without constraints on valuations).

A hyperconnexive (non-normal) conditional **CEX** can be however represented in **CεK**F by an embedding XT as follows:[3]

$$A \rhd B \;:=\; A > (A \leftrightarrow B)$$

This shows that **CEX** can be axiomatized over Classical logic as follows:

EA	If $\vdash A \leftrightarrow B$, then $\vdash (A \rhd C) \leftrightarrow (B \rhd C)$
EC	If $\vdash A \leftrightarrow B$, then $\vdash (C \rhd A) \leftrightarrow (C \rhd B)$
WCM	$(A \rhd \top) \to ((A \rhd (B \wedge C)) \to ((A \rhd B) \wedge (A \rhd C)))$
CC	$((A \rhd B) \wedge (A \rhd C)) \to (A \rhd (B \wedge C))$
ID	$A \rhd A$
CEM	$(A \rhd B) \vee (A \rhd \neg B)$
CMOD	$(\neg A \rhd \bot) \to (B \rhd (A \leftrightarrow B))$
WBT	$(A \rhd B) \to \neg(A \rhd \neg B)$

[3] AFAIK, this is the first hyperconnexive conditional based on Classical logic.

A hyperconnexive (non-normal) conditional **CEX** can be however represented in **CεK**F by an embedding XT as follows:

$$A \rhd B := A > (A \leftrightarrow B)$$

This shows that **CEX** can be axiomatized over Classical logic as follows:

EA	If $\vdash A \leftrightarrow B$, then $\vdash (A \rhd C) \leftrightarrow (B \rhd C)$
EC	If $\vdash A \leftrightarrow B$, then $\vdash (C \rhd A) \leftrightarrow (C \rhd B)$
WCM	$(A \rhd \top) \rightarrow ((A \rhd (B \wedge C)) \rightarrow ((A \rhd B) \wedge (A \rhd C)))$[4]
CC	$((A \rhd B) \wedge (A \rhd C)) \rightarrow (A \rhd (B \wedge C))$
ID	$A \rhd A$
CEM	$(A \rhd B) \vee (A \rhd \neg B)$
CMOD	$(\neg A \rhd \bot) \rightarrow (B \rhd (A \leftrightarrow B))$
WBT	$(A \rhd B) \rightarrow \neg(A \rhd \neg B)$

[4] In **CEX**, modalities are defined as $\Diamond A := A \rhd \top$ and $\Box A := \neg A \rhd \bot$.

Notice that any Classical logic including EA, EC, ID and WBT is trivialized when adding the stronger CM axiom (Weiss, 2019):

CM $(A \triangleright (B \wedge C)) \rightarrow ((A \triangleright B) \wedge (A \triangleright C))$

Surprisingly, **CεK**F can be gained back by an analogous embedding LT:

$$A > B \; := \; A \triangleright (A \leftrightarrow B)$$

As a result, the two conditionals $\triangleright$ and $>$ are interdefinable in **CεK**F and **CEX** resp., and the two systems mutually embeddable:

Theorem

- $\vDash_{\mathbf{CEX}} A$ *iff* $\vDash_{\mathbf{C\varepsilon K}\mathsf{F}} \mathtt{XT}(A)$
- $\vDash_{\mathbf{C\varepsilon K}\mathsf{F}} A$ *iff* $\vDash_{\mathbf{CEX}} \mathtt{LT}(A)$

I reviewed Epsilon Calculus and characterized a notion of choice-functional reasoning based on its expressive properties.

Then, I showed how the semantics machinery underlying epsilon terms' interpretation can be adapted to intensional logics.

The resulting Epsilon Modal logics allow for choice-functional reasoning. I made the claim more precise by showing that the two systems are mutually embeddable.

Finally, I showed how EML models can be represented in Chellas' frames and how an hyperconnexive conditional based on Classical logic can be embedded.

The expressive power of EMLs and its possible further extensions and characterizations, however, have yet to be fully explored.

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