Maximally Substructural Classical Logic

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- Some logical principles explicitly mention logical constants (conjunction, negation, etc.); others do not.
- The former are called *operational*; the latter are called *structural*.

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$$\text{Id}\ \frac{}{A \mathrel{-\!\!\!3} A}$$

$$\times\text{Cut}\ \frac{\Gamma \mathrel{-\!\!\!3} \Delta, A \qquad A, \Sigma \mathrel{-\!\!\!3} \Pi}{\Gamma, \Sigma \mathrel{-\!\!\!3} \Delta, \Pi} \qquad +\text{Cut}\ \frac{\Gamma \mathrel{-\!\!\!3} \Delta, A \qquad A, \Gamma \mathrel{-\!\!\!3} \Delta}{\Gamma \mathrel{-\!\!\!3} \Delta}$$

$$\text{C}{\mathrel{-\!\!\!3}}\ \frac{A, A, \Gamma \mathrel{-\!\!\!3} \Delta}{A, \Gamma \mathrel{-\!\!\!3} \Delta} \qquad {\mathrel{-\!\!\!3}}\text{C}\ \frac{\Gamma \mathrel{-\!\!\!3} \Delta, A, A}{\Gamma \mathrel{-\!\!\!3} \Delta, A}$$

$$\text{W}{\mathrel{-\!\!\!3}}\ \frac{\Gamma \mathrel{-\!\!\!3} \Delta}{A, \Gamma \mathrel{-\!\!\!3} \Delta} \qquad {\mathrel{-\!\!\!3}}\text{W}\ \frac{\Gamma \mathrel{-\!\!\!3} \Delta}{\Gamma \mathrel{-\!\!\!3} \Delta, A}$$

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$$\text{C}{\ni}\ \frac{A, A, \Gamma \ni \Delta}{A, \Gamma \ni \Delta} \qquad {\ni}\text{C}\ \frac{\Gamma \ni \Delta, A, A}{\Gamma \ni \Delta, A}$$

$$\text{W}{\ni}\ \frac{\Gamma \ni \Delta}{A, \Gamma \ni \Delta} \qquad {\ni}\text{W}\ \frac{\Gamma \ni \Delta}{\Gamma \ni \Delta, A}$$

- Here, $A, B, \ldots$ are formulas of the relevant object language, $\Gamma, \Delta, \ldots$ are multisets of formulas, and $\ni$ represents logical consequence.

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- Here, $A, B, \ldots$ are formulas of the relevant object language, $\Gamma, \Delta, \ldots$ are multisets of formulas, and $\mathrel{-\!\!\!3}$ represents logical consequence.
- 'C' stands for 'Contraction', 'W' for 'Weakening', 'Id' for 'Identity', '+Cut' for 'Additive Cut' and '×Cut' for 'Multiplicative Cut'.

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- **Proof theory**
  - *Weaker sense:* ADMISSIBILITY. Whenever the premises are provable, the conclusion is provable.
  - *Stronger sense:* DERIVABILITY. Whenever we add the premises as axioms, the conclusion becomes provable.

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- **Proof theory**
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  - *Stronger sense:* DERIVABILITY. Whenever we add the premises as axioms, the conclusion becomes provable.
- **Model theory**
  - *Weaker sense:* GLOBAL VALIDITY. Whenever the premises are valid, the conclusion is valid.
  - *stronger sense:* LOCAL VALIDITY. Whenever the premises are satisfied by an interpretation, the conclusion is satisfied by that interpretation.

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- Its model-theoretic presentation using Boolean bivalued interpretations makes the structural principles both globally and locally valid.
- In some standard sequent calculi for classical logic, such as Gentzen's LK [2], the structural principles are both admissible and derivable.
- That's why classical logic is usually regarded as a paradigmatic example of a *structural* logical theory.

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- In proof theory, this amounts to having sequent calculi for classical logic where some structural rules are admissible but not derivable.
- Such systems have been intensively studied, because of various proof-theoretical virtues they exhibit (e.g. easy proof-search!)

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- Then, Rosenblatt [7] defined a consequence relation coextensive with classical logic that invalidates not only Cut but also Contraction.

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- For instance, Girard [3] and later Cobreros et. al. [1] and Ripley [6] defined consequence relations coextensive with classical logic that locally invalidate Cut.
- Then, Rosenblatt [7] defined a consequence relation coextensive with classical logic that invalidates not only Cut but also Contraction.
- These consequence relations are interesting, because they allow us to model various (possibly paradoxical!) phenomena with theories that are *substructural*, but conservatively extend classical logic.

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- The purpose of this investigation, then, is to start filling this gap in the model-theoretic analysis of 'substructural versions' of classical logic.

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- First, we will show how to define a consequence relation that is coextensive with classical logic, but locally invalidates Cut, Contraction and Weakening. Indeed, will show that *any* Tarskian consequence relation has a counterpart that locally invalidates all these principles.
- Then, we will give the first steps towards providing a semantics for the set-based sequent calculus K for classical logic (see, e.g. Indrzejczak [4]), where contraction is built-in in the structure of the sequents, but Weakening is only admissible.

Plan

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1. Maximally Substructural Classical Logic
    - No-Weakening
    - No-Contraction
    - Combining
    - Generalising
2. Towards a semantics for K

No-Weakening

- We introduce a system we call **nwCL**, for ‘No-Weakening Classical Logic’.

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- The system is defined by means of a four valued non-deterministic semantics, and a sui generis notions of consequence.

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- Our propositional language $\mathcal{L}$ has primitive constants $\neg$, $\vee$ and $\wedge$.

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- The set of values is $\{1, 0, 1^\star, 0^\star\}$.
- Our semantics is based on the following non deterministic tables, where $\mathbf{1} = \{1, 1^\star\}$ and $\mathbf{0} = \{0, 0^\star\}$:

	$\neg$
1	$\mathbf{0}$
$1^\star$	$\mathbf{0}$
$0^\star$	$\mathbf{1}$
0	$\mathbf{1}$

$\wedge$	1	$1^\star$	$0^\star$	0
1	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{0}$	$\mathbf{0}$
$1^\star$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{0}$	$\mathbf{0}$
$0^\star$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$
0	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$

$\vee$	1	$1^\star$	$0^\star$	0
1	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$
$1^\star$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$
$0^\star$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{0}$	$\mathbf{0}$
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$1^\star$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{0}$	$\mathbf{0}$
$0^\star$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$
0	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$

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1	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$
$1^\star$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$
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0	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{0}$	$\mathbf{0}$

- A $\star$-*valuation* is any function $v : \mathcal{L} \to \{1, 0, 1^\star, 0^\star\}$ satisfying these tables (it doesn't need to be schematic!). We let $V^\star$ be the set of all $\star$-valuations.

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- As is apparent, values $1^\star$ and $0^\star$ behave as an additional copies of values 1 and 0, respectively.
- While the latter will ensure that all classical counterexamples are available, the former will give us counterexamples to weakening.

No-Weakening

- Given a multiset of formulas Σ, we write $|\Sigma|$ for its root set, and $v(\Sigma)$ for the set $\{v(A) : A \in |\Sigma|\}$.

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- Lastly, $\Sigma \sqcup \Pi$ is the multiset where each formula appears the amount of times it appears in Σ plus the amount of times it appears in Π.

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Definition

$\Gamma \models_{\mathbf{nwCL}} \Delta$ just in case, for every v in $V^{\star}$, it is not the case that the following conditions are all met: (i) $v(\Gamma) \subseteq \mathbf{1}$, (ii) $v(\Delta) \subseteq \mathbf{0}$, and (iii) $(\Gamma \sqcup \Delta)^{\star}_{v} \neq 1$.

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- Intuitively, $1^{\star}$ and $0^{\star}$ only contribute to generate a counterexample when they appear at least twice in the argument.

No-Weakening

- It is easy to see that the system locally invalidates $\ni$W and W$\ni$. Consider the instances

$$\frac{p \ni}{q, p \ni} \qquad\qquad \frac{\ni r}{\ni r, s}$$

They are counterexemplified at any v such that $v(p) = v(q) = 1^{\star}$, and at any v such that $v(r) = v(s) = 0^{\star}$, respectively.

No-Weakening

- It is easy to see that the system locally invalidates $-\!\!3$W and W$-\!\!3$. Consider the instances

  $$\frac{p \mathrel{-\!\!3}}{q, p \mathrel{-\!\!3}} \qquad\qquad \frac{\mathrel{-\!\!3} r}{\mathrel{-\!\!3} r, s}$$

  They are counterexemplified at any v such that $v(p) = v(q) = 1^\star$, and at any v such that $v(r) = v(s) = 0^\star$, respectively.

- But the system also locally invalidates $+$Cut and $\times$Cut, as

  $$\frac{\mathrel{-\!\!3} p \qquad p \mathrel{-\!\!3}}{\mathrel{-\!\!3}}$$

  is an instance of both principles, and is counterexemplified at any v such that $v(p) \in \{1^\star, 0^\star\}$.

No-Weakening

- Let $\models_{\mathbf{CL}}$ stand for the consequence relation of multiple-conclusion classical logic—defined as usual by means of Boolean bivaluations.

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Fact

$\Gamma \models_{\mathbf{nwCL}} \Delta$ just in case $\Gamma \models_{\mathbf{CL}} \Delta$.

No-Contraction

- We dualize the previous trick to define a system we call **ncCL**, for ‘No-Contraction Classical Logic’.

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- The semantics for the system is exactly as before; for notational convenience, we just replace the star '$\star$' by a circle '$\circ$'.

- So, the set of values is $\{1, 0, 1^\circ, 0^\circ\}$, $\mathbf{1} = \{1, 1^\circ\}$, $\mathbf{0} = \{0, 0^\circ\}$, we operate with $\circ$-valuations, and V° is the set of all of them.

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Definition

$\Gamma \models_{\mathbf{ncCL}} \Delta$ just in case, for every v in V°, it is not the case that the following conditions are all met: (i) $v(\Gamma) \subseteq \mathbf{1}$, (ii) $v(\Delta) \subseteq \mathbf{0}$, and (iii) $(\Gamma \sqcup \Delta)^\circ_v \leq 1$.

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- Intuitively, 1° and 0° only contribute to generate a counterexample when they appear exactly once in the argument.

No-Contraction

- It is easy to see that the system locally invalidates $\strictif$C and C$\strictif$. Consider the instances

  $$\frac{p, p \strictif}{p \strictif} \qquad\qquad \frac{\strictif q, q}{\strictif q}$$

  They are counterexemplified at any v such that $v(p) = 1^{\circ}$ and at any v such that $v(q) = 0^{\circ}$, respectively.

No-Contraction

- It is easy to see that the system locally invalidates $\mathrel{-\!\!3}$C and C$\mathrel{-\!\!3}$. Consider the instances

$$\frac{p, p \mathrel{-\!\!3}}{p \mathrel{-\!\!3}} \qquad\qquad \frac{\mathrel{-\!\!3} q, q}{\mathrel{-\!\!3} q}$$

  They are counterexemplified at any v such that $v(p) = 1^\circ$ and at any v such that $v(q) = 0^\circ$, respectively.

- But the system also locally invalidates the principles of Cut, as

$$\frac{r \mathrel{-\!\!3} p \qquad p \mathrel{-\!\!3} s}{r \mathrel{-\!\!3} s} \qquad\qquad \frac{r \mathrel{-\!\!3} s, p \qquad p, r \mathrel{-\!\!3} s}{r \mathrel{-\!\!3} s}$$

  are instances of $\times$Cut and $+$Cut, respectively, and they are counterexemplified at any v such that $v(r) = 1$, $v(s) = 0$ and $v(p) \in \{1^\circ, 0^\circ\}$.

No-Contraction

Fact

$\Gamma \models_{\mathbf{ncCL}} \Delta$ just in case $\Gamma \models_{\mathbf{CL}} \Delta$.

Combining

- Lastly, we define a system we call **msCL**, for 'Maximally Substructural Classical Logic'. The listener might guess how it goes...

Combining

- The set of values is $\{1, 0, 1^\star, 0^\star, 1^\circ, 0^\circ\}$. Letting $\mathbf{1} = \{1, 1^\star, 1^\circ\}$ and $\mathbf{0} = \{0, 0^\star, 0^\circ\}$, the tables are:

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	$\neg$
1	**0**
$1^\star$	**0**
1°	**0**
0°	**1**
$0^\star$	**1**
0	**1**

$\wedge$	1	$1^\star$	1°	0°	$0^\star$	0
1	**1**	**1**	**1**	**0**	**0**	**0**
$1^\star$	**1**	**1**	**1**	**0**	**0**	**0**
1°	**1**	**1**	**1**	**0**	**0**	**0**
0°	**0**	**0**	**0**	**0**	**0**	**0**
$0^\star$	**0**	**0**	**0**	**0**	**0**	**0**
0	**0**	**0**	**0**	**0**	**0**	**0**

$\vee$	1	$1^\star$	1°	0°	$0^\star$	0
1	**1**	**1**	**1**	**1**	**1**	**1**
$1^\star$	**1**	**1**	**1**	**1**	**1**	**1**
1°	**1**	**1**	**1**	**1**	**1**	**1**
0°	**1**	**1**	**1**	**0**	**0**	**0**
$0^\star$	**1**	**1**	**1**	**0**	**0**	**0**
0	**1**	**1**	**1**	**0**	**0**	**0**

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- The set of values is $\{1, 0, 1^\star, 0^\star, 1^\circ, 0^\circ\}$. Letting $\mathbf{1} = \{1, 1^\star, 1^\circ\}$ and $\mathbf{0} = \{0, 0^\star, 0^\circ\}$, the tables are:

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1	**0**
$1^\star$	**0**
1°	**0**
0°	**1**
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1	**1**	**1**	**1**	**0**	**0**	**0**
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1°	**1**	**1**	**1**	**0**	**0**	**0**
0°	**0**	**0**	**0**	**0**	**0**	**0**
$0^\star$	**0**	**0**	**0**	**0**	**0**	**0**
0	**0**	**0**	**0**	**0**	**0**	**0**

$\vee$	1	$1^\star$	1°	0°	$0^\star$	0
1	**1**	**1**	**1**	**1**	**1**	**1**
$1^\star$	**1**	**1**	**1**	**1**	**1**	**1**
1°	**1**	**1**	**1**	**1**	**1**	**1**
0°	**1**	**1**	**1**	**0**	**0**	**0**
$0^\star$	**1**	**1**	**1**	**0**	**0**	**0**
0	**1**	**1**	**1**	**0**	**0**	**0**

- A $\star\circ$-*valuation* is a function $v : \mathcal{L} \to \{1, 0, 1^\star, 0^\star, 1^\circ, 0^\circ\}$ satisfying the above tables, and $V^{\star\circ}$ is the set of all such valuations.

Combining

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$\Gamma \models_{\mathbf{msCL}} \Delta$ just in case, for every v in $V^{\star\circ}$, it is not the case that all the following hold: (i) $v(\Gamma) \subseteq \mathbf{1}$, (ii) $v(\Delta) \subseteq \mathbf{0}$, (iii) $(\Gamma \sqcup \Delta)^{\star}_{v} \neq 1$, and (iv) $(\Gamma \sqcup \Delta)^{\circ}_{v} \leq 1$

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$\Gamma \models_{\mathbf{msCL}} \Delta$ just in case, for every v in $V^{\star\circ}$, it is not the case that all the following hold: (i) $v(\Gamma) \subseteq \mathbf{1}$, (ii) $v(\Delta) \subseteq \mathbf{0}$, (iii) $(\Gamma \sqcup \Delta)^{\star}_{v} \neq 1$, and (iv) $(\Gamma \sqcup \Delta)^{\circ}_{v} \leq 1$

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- All the previous local counterexamples to structural principles are still available.
- This means that, of all the structural principles we have considered, **msCL** only locally validates Id.

Fact

$\Gamma \models_{\mathbf{msCL}} \Delta$ just in case $\Gamma \models_{\mathbf{CL}} \Delta$

Generalising

- Let $\mathcal{L}$ be *any* propositional language.
- Let $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ be a logical matrix of type $\mathcal{L}$, with $\{0, 1\} \subseteq \mathcal{V}$, $1 \in \mathcal{D}$ and $0 \notin \mathcal{D}$.
- We define the non-deterministic matrix $\mathcal{M}^{\star\circ} = \langle \mathcal{V}^{\star\circ}, \mathcal{D}^{\star\circ}, \mathcal{O}^{\star\circ} \rangle$.

  $\mathcal{V}^{\star\circ} = \mathcal{V} \cup \{1^*, 0^*, 1^\circ, 0^\circ\}$.

  $\mathcal{D}^{\star\circ} = \mathcal{D} \cup \{1^*, 1^\circ\}$.

  For each n-ary operation $\#$ in $\mathcal{O}$, $\mathcal{O}^{\star\circ}$ contains the n-ary operation $\#^*$ defined as follows:

$$\#^*(x_1, \ldots, x_n) = \begin{cases} \{1, 1^*, 1^\circ\} & \text{if } x_1, \ldots, x_n \in \mathcal{V} \text{ and } \#(x_1, \ldots, x_n) = 1 \\ \{0, 0^*, 0^\circ\} & \text{if } x_1, \ldots, x_n \in \mathcal{V} \text{ and } \#(x_1, \ldots, x_n) = 0 \\ \{y\} & \text{if } x_1, \ldots, x_n \in \mathcal{V} \text{ and } \#(x_1, \ldots, x_n) = y, \text{ with } 0 \neq y \neq 1 \\ \#^*(!(x_1, \ldots, x_n)) & \text{if } x_i \notin \mathcal{V} \text{ for some } 1 \leq i \leq n \end{cases}$$

  where $!(x_1, \ldots, x_n)$ is the n-tuple that results from replacing in $x_1, \ldots, x_n$ each 1^* and 1° with 1 and each 0^* and 0° with 0.

Generalising

- The concept of an $\mathcal{M}^{\star\circ}$-*valuation* for $\mathcal{L}$ is defined as usual for non-deterministic matrices.
- Now, we define the logic induced by $\mathcal{M}^{\star\circ}$:

Definition

$\Gamma \models_{\mathcal{M}^{\star\circ}} \Delta$ just in case, for every $\mathcal{M}^{\star\circ}$-valuation v, it is not the case that the following hold: (i) $v(\Gamma) \subseteq \mathcal{D}^{\star\circ}$, (ii) $v(\Delta) \subseteq \mathcal{V}^{\star\circ}/\mathcal{D}^{\star\circ}$, (iii) $(\Gamma \sqcup \Delta)^{\star}_{v} \neq 1$, and (iv) $(\Gamma \sqcup \Delta)^{\circ}_{v} \leq 1$

- It is easy to check that this logic locally invalidates Contraction, Weakening and Cut.
- Now, let $\models_{\mathcal{M}}$ be the logic induced by the logical matrix $\mathcal{M}$ in the usual way.

Fact

$\Gamma \models_{\mathcal{M}} \Delta$ just in case $\Gamma \models_{\mathcal{M}^{\star\circ}} \Delta$

Generalising

- A logic **L** is *Tarskian* just in case its consequence relation $\models_{\mathbf{L}}$ is closed under (that is, globally validates) Id, Weakening and Cut.

- Roughly, Wójcicki [10] proved the following: for any Tarskian logic **L**, there is a class $\mathbb{M}$ of logical matrices such that $\models_{\mathbf{L}}$ and $\models_{\mathbb{M}}$ validate the same arguments. (Here, $\mathbb{M}$ is possibly infinite, and for each of the matrices it contains, its universe is also possibly infinite.)

- Now, applying this to our previous result, we obtain: *for each Tarskian logic* **L**, *there is a class* $\mathbb{M}^{\star\circ}$ *of matrices such that* $\models_{\mathbf{L}}$ *and* $\models_{\mathbb{M}^{\star\circ}}$ *validate the same arguments, but* $\models_{\mathbb{M}^{\star\circ}}$ *locally invalidates Contraction, Weakening and Cut.*

- This can be seen as a strong generalisation of a result by Szmuc [8], who showed (also appealing to Wójcicki's theorem) that any Tarskian logic has a coextensive counterpart that locally invalidates Cut.

Plan

1. Maximally Substructural Classical Logic
    - No-Weakening
    - No-Contraction
    - Combining
    - Generalising

2. Towards a semantics for K

Towards a semantics for K

- A *sequent* is a pair $\langle \Gamma, \Delta \rangle$ where Γ and Δ are (names of) collections of formulas.

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- We denote sequent $\langle \Gamma, \Delta \rangle$ as $\Gamma \Rightarrow \Delta$.
- Let **L** be a logic defined by model-theoretic means, and v one of its admissible valuations. A sequent $\Gamma \Rightarrow \Delta$ is **L**-*satisfied* at v just in case v is not a counterexample to the claim $\Gamma \models_{\mathbf{L}} \Delta$.

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- A sequent $\Gamma \Rightarrow \Delta$ is *valid* in **L** just in case $\Gamma \models_{\mathbf{L}} \Delta$.

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- A *metasequent* is a pair $\langle \mathfrak{A}, \mathfrak{b} \rangle$ where $\mathfrak{A} \cup \{\mathfrak{b}\}$ is a set of sequents. We write metasequents in rule form:

$$\frac{\Gamma_1 \Rightarrow \Delta_1 \qquad \dots \qquad \Gamma_n \Rightarrow \Delta_n}{\Sigma \Rightarrow \Pi}$$

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- So, the *rules* of a sequent calculus are just schematic metasequents.
- A metasequent $\langle \mathfrak{A}, \mathfrak{b} \rangle$ is *locally valid* in a logic **L** just in case, for every admissible valuation, if the sequents in $\mathfrak{A}$ are all satisfied, $\mathfrak{b}$ is satisfied.

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  - *Weak adequacy:* A logic **L** such that a sequent $\Gamma \Rightarrow \Delta$ is provable in $\mathcal{S}$ just in case $\Gamma \models_{\mathbf{L}} \Delta$.
  - *Strong adequacy:* A logic **L** such that a metasequent $\langle \mathfrak{A}, \mathfrak{b} \rangle$ is derivable in $\mathcal{S}$ just in case it is locally valid in **L**.

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They are instances of the multiplicative and the additive left rule for $\wedge$, respectively. They are both counterexemplified at any valuation v such that $v(p) = v(q) = 1$ and $v(r) = 1^\star = v(p \wedge q)$.

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- Thus, e.g. the one on the left is not a sequent, but the one on the right is

$$p \vee q, p, p \vee q \Rightarrow q \qquad\qquad p \vee q, p \Rightarrow q$$

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$$p \vee q, p, p \vee q \Rightarrow q \qquad\qquad p \vee q, p \Rightarrow q$$

- We understand sequents in this way to avoid various syntactical issues indicated by Negri and von Plato [9]

Towards a semantics for K

- The rules of K

$$\text{Id}\ \frac{}{A, \Gamma \Rightarrow A}$$

$$\text{L}\vee\ \frac{A, \Gamma \Rightarrow \Delta \qquad B, \Gamma \Rightarrow \Delta}{A \vee B, \Gamma \Rightarrow \Delta} \qquad \text{R}\vee\ \frac{\Gamma \Rightarrow \Delta, A, B}{\Gamma \Rightarrow A \vee B}$$

$$\text{L}\wedge\ \frac{A, B, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta} \qquad \text{R}\wedge\ \frac{\Gamma \Rightarrow \Delta, A \qquad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \wedge B}$$

$$\text{L}\neg\ \frac{\Gamma \Rightarrow \Delta, A}{\neg A, \Gamma \Rightarrow \Delta} \qquad \text{R}\neg\ \frac{A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg A}$$

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- A strong semantics for this system would have to locally validate contraction (imposed by the structure of the sequents) but invalidate Weakening and Cut.

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	$\neg$
1	0
$1^\star$	$0^\star$
$0^\star$	$1^\star$
0	1

$\wedge$	1	$1^\star$	$0^\star$	0
1	1	$1^\star$	$0^\star$	0
$1^\star$	$1^\star$	$1^\star$	$0^\star$	$0^\star$
$0^\star$	$0^\star$	$0^\star$	$0^\star$	$0^\star$
0	0	$0^\star$	$0^\star$	0

$\vee$	1	$1^\star$	$0^\star$	0
1	1	$1^\star$	$1^\star$	1
$1^\star$	$1^\star$	$1^\star$	$1^\star$	$1^\star$
$0^\star$	$1^\star$	$1^\star$	$0^\star$	$0^\star$
0	1	$1^\star$	$0^\star$	0

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$0^\star$	$0^\star$	$0^\star$	$0^\star$	$0^\star$
0	0	$0^\star$	$0^\star$	0

$\vee$	1	$1^\star$	$0^\star$	0
1	1	$1^\star$	$1^\star$	1
$1^\star$	$1^\star$	$1^\star$	$1^\star$	$1^\star$
$0^\star$	$1^\star$	$1^\star$	$0^\star$	$0^\star$
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- The stared values behave in an ‘infectious’ way: whenever some subformula of A has a stared value, A has a stared value.

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1	1	$1^\star$	$1^\star$	1
$1^\star$	$1^\star$	$1^\star$	$1^\star$	$1^\star$
$0^\star$	$1^\star$	$1^\star$	$0^\star$	$0^\star$
0	1	$1^\star$	$0^\star$	0

- The stared values behave in an ‘infectious’ way: whenever some subformula of A has a stared value, A has a stared value.

- The tables are *normal*: classical inputs deliver the expected classical output.

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- We add an additional value $^1/_2$ and make the tables non-deterministic again. Let $\mathbb{1} = \{1, {}^1/_2\}$, $\mathbb{1}^\star = \{1^\star, {}^1/_2\}$, and similarly for $\mathbb{0}$ and $\mathbb{0}^\star$

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$\wedge$	1	$1^\star$	$1/2$	$0^\star$	0
1	$\mathbb{1}$	$\mathbb{1}^\star$	$\{1/2\}$	$\mathbb{0}^\star$	$\mathbb{0}$
$1^\star$	$\mathbb{1}^\star$	$\mathbb{1}^\star$	$\{1/2\}$	$\mathbb{0}^\star$	$\mathbb{0}^\star$
$1/2$	$\{1/2\}$	$\{1/2\}$	$\{1/2\}$	$\mathbb{0}^\star$	$\mathbb{0}$
$0^\star$	$\mathbb{0}^\star$	$\mathbb{0}^\star$	$\mathbb{0}^\star$	$\mathbb{0}^\star$	$\mathbb{0}^\star$
0	$\mathbb{0}$	$\mathbb{0}^\star$	$\mathbb{0}$	$\mathbb{0}^\star$	$\mathbb{0}$

$\vee$	1	$1^\star$	$1/2$	$0^\star$	0
1	$\mathbb{1}$	$\mathbb{1}^\star$	$\mathbb{1}$	$\mathbb{1}^\star$	$\mathbb{1}$
$1^\star$	$\mathbb{1}^\star$	$\mathbb{1}^\star$	$\mathbb{1}^\star$	$\mathbb{1}^\star$	$\mathbb{1}^\star$
$1/2$	$\mathbb{1}$	$\mathbb{1}^\star$	$\{1/2\}$	$\{1/2\}$	$\{1/2\}$
$0^\star$	$\mathbb{1}^\star$	$\mathbb{1}^\star$	$\{1/2\}$	$\mathbb{0}^\star$	$\mathbb{0}^\star$
0	$\mathbb{1}$	$\mathbb{1}^\star$	$\{1/2\}$	$\mathbb{0}^\star$	$\mathbb{0}$

	$\neg$
1	$\mathbb{0}$
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$1/2$	$\{1/2\}$
$0^\star$	$\mathbb{1}^\star$
0	$\mathbb{1}$

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Definition

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- So, the definition of consequence is exactly as in **nwCL**: we only changed the set of valuations which we quantify over.

Towards a Semantics for K

- Just like **nwCL**, logic **nwCLG** locally invalidates Cut and Weakening.

Towards a Semantics for K

- Just like **nwCL**, logic **nwCLG** locally invalidates Cut and Weakening.
- Unlike **nwCL** (which locally invalidates all rules of K), logic **nwCLG** allows to proof strong soundness:

Fact

Every metasequent derivable in K is locally valid in **nwCLG**

Towards a Semantics for K

- What is more, **nwCLG** locally invalidates a number of rules for the connectives that distinguish K from other systems.

Towards a Semantics for K

- What is more, **nwCLG** locally invalidates a number of rules for the connectives that distinguish K from other systems.
- For one thing, it locally invalidates all the non-invertible rules for the connectives:

$$\frac{A, \Gamma \Rightarrow \Delta \qquad B, \Sigma \Rightarrow \Pi}{A \vee B, \Gamma, \Sigma \Rightarrow \Delta, \Pi} \qquad \frac{\Gamma \Rightarrow \Delta, A/B}{\Gamma \Rightarrow A \vee B}$$

$$\frac{A/B, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta, A \qquad \Sigma \Rightarrow \Pi, B}{\Gamma, \Sigma \Rightarrow \Delta, \Pi, A \wedge B}$$

Towards a semantics for K

- For another thing, it invalidates the *inverses* of all the rules of K, which are of course not derivable in the system:

$$\frac{A \vee B, \Gamma \Rightarrow \Delta}{A/B, \Gamma \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta, A \vee B}{\Gamma \Rightarrow, A, B}$$

$$\frac{A \wedge B, \Gamma \Rightarrow \Delta}{A, B, \Gamma \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta, A \wedge B}{\Gamma \Rightarrow \Delta, A/B}$$

$$\frac{\neg A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, A} \qquad \frac{\neg A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, A}$$

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- Thus, while **nwCLG** brings us closer to a semantics for K, it is still only an approximation.

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Taking stock

- We defined versions of classical logic that invalidate the principles of Contraction, Cut and Weakening.
- We showed that, given any Tarskian logic **L**, there is a 'maximally substructural' logic **msL** violating all those principles.
- We gave some steps towards modifying our systems to provide a semantics to the well known calculus for classical logic K, where contraction is implicit, and Weakening and Cut are merely admissible.

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Thanks!!