Notational Variance in Substructural Logics

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Introduction

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- A *logical system* is just a formal system whose corresponding relation is understood as logical consequence.

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- Often, two or more logical systems differ only in a superficial way:
  - In the choice of symbols (‘$\wedge$’ vs. ‘&’ for conjunction)
  - In the choice of syntactic conventions (infix vs. prefix notation)
  - In the choice of primitive operations ($\{\wedge, \neg\}$ vs. $\{\vee, \neg\}$)

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  - In the choice of syntactic conventions (infix vs. prefix notation)
  - In the choice of primitive operations ($\{\wedge, \neg\}$ vs. $\{\vee, \neg\}$)

- When two logical systems differ only in a superficial way, we say that they are *notational variants* of each other.

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- *Guiding question*: What conditions are necessary and/or sufficient to say that two logical systems are notational variants?

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- Lately, we have seen the emergence of systems that are substructural in a radical sense: they abandon the reflexivity and/or transitivity of entailment.
- I will claim that the usual criteria of notational variance fail in presence of radically substructural logical systems.
- Then, I will give the first steps towards providing a new, improved criterion.

Plan

1. The Standard Approach
2. The Non-Reflexive Challenge
3. The Non-Transitive Challenge
4. Taking Stock

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The Standard Approach

- The approach can be found for instance in Caleiro & Gonçalves [5], Frech [13], Kocurek [15], Pelletier & Urquhart [18].

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- Central idea: *Two logical systems are notational variants if and only if, once we translate them properly, they validate the same arguments.*
- For concreteness, I will focus on French's precisification of this idea. But my main points are applicable to the other versions as well.

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- A translation τ is *compositional* iff for each n-ary constant $\#$ of $\mathcal{L}_1$ there is a (perhaps defined) n-ary connective $\#^\tau$ of $\mathcal{L}_2$ such that, for every $A_1, ..., A_n \in \mathcal{L}_1$, $\tau(\#(A_1, ..., A_n) = \#^\tau(\tau(A_1), ..., \tau(A_n))$.

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- A translation τ is *definitional* iff it is both variable-fixed and compositional.
- A translation τ *faithfully embeds* a logic $\mathbf{L}_1$ in a logic $\mathbf{L}_2$ just in case:

$$A \dashv\vdash_1 B \quad \text{if and only if} \quad \tau(A) \dashv\vdash_2 \tau(B)$$

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- This is what we will call Simple Extensional Approach:

  (SEA) Two logical systems $\mathbf{L_1}$ and $\mathbf{L_2}$ are notational variants just in case there are definitional translations τ_1 and τ_2 that faithfully embed the former in the latter and viceversa, and in addition

$$A \dashv\vdash_1 \tau_2(\tau_1(A)) \qquad\qquad A \dashv\vdash_2 \tau_1(\tau_2(A))$$

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- But my points also apply to other non-reflexive systems, such as Humberstone's heterogeneous logic [14] or Correia's logic of grounding [8].

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- $\mathcal{L}$ is a sentential language with primitive constants $\neg$, $\vee$ and $\wedge$.
- Relation $\dashv\vdash_{\mathbf{TS}}$ is defined using the *strong Kleene* valuations:

$\neg$	A
1	0
1/2	1/2
0	1

$\vee$	1	1/2	0
1	1	1	1
1/2	1	1/2	1/2
0	1	1/2	0

$\wedge$	1	1/2	0
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- Label '**TS**' stands for 'Tolerant-Strict'.
- Any v such that $v(p) = 1/2$ shows that $p \not\dashv\!\vDash_{\mathbf{TS}} p$.

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- Clearly, τ_1 and τ_2 are definitional translations which faithfully embed **TS** into itself.
- However, inversion fails. For instance, since $\tau_2(\tau_1(p)) = p$, we have that $p \not\dashv\vdash_{\mathbf{TS}} \tau_2(\tau_1(p))$.
- So, the standard approach clearly undergenerates. Indeed, it delivers that some logical systems are not notational variants of themselves!

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- Also, note that we are not asking that τ be definitional anymore.

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- To show that they are notational variants, it seems natural to take the pair of translations τ_1 and τ_2 defined as follows:

$$\tau_1(p_i) = p_i \qquad \tau_2(p_i) = p_i$$

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- But these translations are clearly not inverses of each other, viz. $\tau_1^{-1} \neq \tau_2$
- According to $(*)$, we need to provide some weird-looking, non-compositional translation to show that $\mathbf{CL_1}$ and $\mathbf{CL_2}$ are notational variants

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- So, two logical systems would be notational variants if and only if they satisfy either (SEA) or ($*$).
- Setting the *ad hoc* flavour of this option aside, the resulting criterion is still not satisfactory.
- It rules out some well-behaved translations between presentations of **TS** with different primitive constants.

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- Extensional Approach Debugged:

  (EAD) Two logical systems $\mathbf{L_1}$ and $\mathbf{L_2}$ are notational variants just in case there are definitional translations τ_1 and τ_2 that faithfully embed the former in the latter and viceversa, and in addition

$$
\begin{array}{rcl}
A, \Gamma \dashv_1 C & \text{if and only if} & \tau_2(\tau_1(A)), \Gamma \dashv_1 C \\
\Gamma \dashv_1 C & \text{if and only if} & \Gamma \dashv_1 \tau_2(\tau_1(C)) \\
A, \Gamma \dashv_2 C & \text{if and only if} & \tau_1(\tau_2(A)), \Gamma \dashv_2 C \\
\Gamma \dashv_2 C & \text{if and only if} & \Gamma \dashv_2 \tau_1(\tau_2(C))
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- Also, whenever two systems are reflexive and transitive, the old criterion and the new one collapse: they are satisfied or unsatisfied together.
- That is, the new criterion gives the right answers in the familiar cases.
- Thus, I take (EAD) to be an improvement over (SEA).
- When two logical satisfy (EAD), I will say that they are *coextensive modulo translation*.

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- **ST** is quite unique in that it can be regarded as a non-transitive counterpart of classical logic.

- But the underlying phenomenon is quite general: there is a wide range of systems having an **ST**-like counterpart (Fitting [11], Szmuc [23]).

The Non-Transitive Challenge

- We present **ST** with the system $\langle \mathcal{L}, \dashv_{\textbf{ST}} \rangle$

  $\Gamma \dashv_{\textbf{ST}} B$ just in case, for every strong Kleene valuation v, if $v(\Gamma) \subseteq \{1\}$ then $v(B) \in \{1, {}^{1}/_{2}\}$.

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- Label '**ST**' stands for 'Strict-Tolerant'.

- Let $v(p) = 1$, $v(q) = 1/2$ and $v(r) = 0$. Then, v is not a counterexample to either $p \dashv_{\mathbf{ST}} q$ or $q \dashv_{\mathbf{ST}} r$, but it is a counterexample to $p \dashv_{\mathbf{ST}} r$.

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- We present **CL** as the system $\langle \mathcal{L}, \Dashv_{\mathbf{CL}} \rangle$, where $\mathcal{L}$ is as before and $\Dashv_{\mathbf{CL}}$ is defined as usual using Boolean bivaluations.

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- Trivially, then, they are declared notational variants by (EAD).
- However, there is some consensus in the literature that **ST** *is not* **CL** (e.g. Barrio et. al. [1], Cobreros et. al. [7], Dicher and Paoli [10]).
- Arguably, the main difference between them stems from their possible *applications*. In the words of Barrio, Pailos and Szmuc (BPS) [3]:

  [T]here is something deeply uncomfortable about such a claim [viz. that **ST** *and* **CL** *are identical]:* **CL** *is prone to trivialization when faced with transparent truth, vague phenomena and much more, while* **ST** *does not fall into such troubles. Hence, it seems that these systems are not identical, even if this in itself does not suggest a criterion to tell them apart.*

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- We need to identify properties that allow us to set **ST** and **CL** apart.
- One promising candidate is metainferential validity.

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- An *inference* is a pair $\langle \Gamma, A \rangle$ where $\Gamma \cup \{A\} \subseteq \mathcal{L}$. We denote it $\Gamma \succ A$.

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- A *metainference* is a pair $\langle \mathfrak{A}, \mathfrak{b}\rangle$, where $\mathfrak{A} \cup \{\mathfrak{b}\}$ is a set of inferences. We usually present metainferences in a rule-like fashion. For instance,

$$\frac{p \succ q \qquad q \succ r}{p \succ r}\ (\star)$$

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- A metainference $\langle \mathfrak{A}, \mathfrak{b} \rangle$ is *valid* in a logic $\mathbf{L}$ iff, for every relevant valuation, if it satisfies all inferences in $\mathfrak{A}$ then it satisfies $\mathfrak{b}$.

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- But the converse is not true—($\star$) is a paradigmatic example.
- An idea: *Two logical systems are notational variants iff, once we translate them properly, they validate the same inferences and metainferences.*

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- Define that a *meta*$_0$*inference* is an inference, and for $n \geq 1$, a *meta*$_{n+1}$*inference* is a pair $\langle \mathfrak{A}, \mathfrak{b} \rangle$ where $\mathfrak{A} \cup \{\mathfrak{b}\}$ is a set of meta$_n$inferences.

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- Satisfaction and validity of meta$_n$inferences with $n \geq 2$ are defined in similar way as at the corresponding notions for $n = 1$.
- Let us relabel **ST** as $\mathbf{ST}_1$.
- BPS show that, for each n, there is a system $\mathbf{ST}_n$ that coincides with **CL** up to and including meta$_n$inferences, but diverges from there upwards.

The Non-Transitive Challenge

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- If having the same meta_0inferences but different meta_1inferences is enough to set two systems apart, then having the same meta_1inferences but different meta_2inferences should be enough as well.
- And the reasoning generalizes...
- The authors conclude that *there is no finite metainferential level that is enough to identify a logical system.*

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- In our context, the proposal can be put roughly as follows:

  (∗∗) Two logical systems are notational variants if and only if there is a pair of translations that makes them coextensive (modulo translation) at the level of inferences and also at the level of meta_ninferences of any $n \geq 1$.

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- Criterion (∗∗) asks us to compare logical systems across an infinite sequence of increasingly complex objects.
- The criterion is immune to the particular problem that affected the standard approach, since it classifies **ST** and **CL** as clearly different systems.
- However, (∗∗) has its own drawbacks.

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- The most obvious drawback is its counterintuitive character.
- It seems plausible to say that we have some informal understanding of the notion of ‘inference’.
- Perhaps even (though to a lesser extent) we have some grasp of what is a ‘metainference’.
- But the nature of metainferences of higher levels, and their link to our inferential practices, is unclear at best.

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- As an example we have the system $\mathbf{ST}_\omega$ defined by Pailos [17]. Very roughly, it results by taking the union of all the $\mathbf{ST}_n$s.
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- As an example we have the system $\mathbf{ST}_\omega$ defined by Pailos [17]. Very roughly, it results by taking the union of all the $\mathbf{ST}_n$s.
- $\mathbf{ST}_\omega$ validates the same things as **CL** at every level. However, the prevailing opinion is that $\mathbf{ST}_\omega$ is not **CL** (Porter [19], Scambler [22]).
- The reason has again to do with applications. While $\mathbf{ST}_\omega$ is compatible with naive truth, **CL** is not. And we can agree with Porter in that

  [T]he consequences of a set of axioms are not presentation-dependent features of a logic; the same axioms should not generate different theories depending on how we present the logic.

The Non-Transitive Challenge

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- So, considering higher and higher metainferential levels is not enough to individuate a logical system.
- Are there other alternatives?

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- In at least one way of understanding the subject matter of logic, logical systems are *tools* that allow us to make valid inferences.
- The debate around **ST** often relies on an implicit assumption: two tools *cannot* be the same if they don't allow you to do the same things.
- I propose to make this idea explicit, and to take it as an adequacy condition for any criterion of notational variance. I call it *indiscernibility under applications*:

  Two logical systems are notational variants only if they deliver the same results when loaded with the same theoretical assumptions.

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- In at least one way of understanding the subject matter of logic, logical systems are *tools* that allow us to make valid inferences.
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- I propose to make this idea explicit, and to take it as an adequacy condition for any criterion of notational variance. I call it *indiscernibility under applications*:

  Two logical systems are notational variants only if they deliver the same results when loaded with the same theoretical assumptions.

- Of course, this is quite informal and vague. The challenge is to formulate a precise criterion which does justice to the intuitive idea behind it.

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- Also, there is a somewhat separate debate about when two logical theories are *equivalent* (Dewar [9], Wigglesworth [24], Williamson [25], Woods [26]).

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- Still in the discussion around notational variance, related adequacy conditions were endorsed by French [13] and Caleiro and Gonçalves [5].
- Also, there is a somewhat separate debate about when two logical theories are *equivalent* (Dewar [9], Wigglesworth [24], Williamson [25], Woods [26]).
- In this debate, all authors assume some adequacy criterion akin to ours.

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- Now, for this to work, we cannot understand a non-logical theory as just a set of sentences closed under consequence
- This would fail to pinpoint the difference between **CL** and **ST**.
- The theoretical closures of **CL** and **ST** are identical, for any sets of axioms whatsoever!

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- So, I propose to understand a non-logical theory as collection of *inferences*.
- In this framework, assuming a *sentence* A as an axiom will be tantamount to assuming the inference $\succ A$.

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- Let $\mathbf{L} = \langle \mathcal{L}, \dashv \rangle$ be a formal system, and $\mathfrak{T}$ be a set of inferences on $\mathcal{L}$.

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- By $\mathbf{L}^{\mathfrak{T}} = \langle \mathcal{L}, \Yleft^{\mathfrak{T}} \rangle$ we denote the formal theory that results from adding all inferences in $\mathfrak{T}$ to $\mathbf{L}$.
- Relation $\Yleft^{\mathfrak{T}}$ might be obtained in different ways. For instance,
  - If $\Yleft$ is given by model-theoretic means, then we can define $\Yleft^{\mathfrak{T}}$ by restricting the models of $\Yleft$ to those that satisfy each inference in $\mathfrak{T}$
  - If $\Yleft$ is given by means of a sequent calculus, then we can define $\Yleft^{\mathfrak{T}}$ as the result of adding each inference in $\mathfrak{T}$ to this calculus.

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  - If $\dashv$ is given by means of a sequent calculus, then we can define $\dashv^{\mathfrak{T}}$ as the result of adding each inference in $\mathfrak{T}$ to this calculus.
- Either way, the informal reading of $\mathbf{L}^{\mathfrak{T}}$ is the same: it is the non-logical theory $\mathfrak{T}$ over $\mathbf{L}$.

The Non-Transitive Challenge

- Intensional Approach:
  (IA) Two logics $\mathbf{L}_1 = \langle \mathcal{L}_1, \multimap_1 \rangle$ and $\mathbf{L}_2 = \langle \mathcal{L}_2, \multimap_2 \rangle$ are notational variants if and only if there is a pair of translations τ_1 and τ_2 such that:
  - For every set of inferences $\mathfrak{T}$ on $\mathcal{L}_1$, τ_1 and τ_2 render coextensive (modulo translation) the theories $\mathbf{L}_1^{\mathfrak{T}}$ and $\mathbf{L}_2^{\tau_1(\mathfrak{T})}$.
  - For every set of inferences $\mathfrak{S}$ on $\mathcal{L}_2$, τ_1 and τ_2 render coextensive (modulo translation) the theories $\mathbf{L}_2^{\mathfrak{S}}$ and $\mathbf{L}_1^{\tau_2(\mathfrak{S})}$.

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  - For every set of inferences $\mathfrak{S}$ on $\mathcal{L}_2$, τ_1 and τ_2 render coextensive (modulo translation) the theories $\mathbf{L}_2^{\mathfrak{S}}$ and $\mathbf{L}_1^{\tau_2(\mathfrak{S})}$.

- *Intuitively*: two logical systems are notational variants if and only if, once we translate them properly, they give rise to theories that are coextensive modulo translation (in our extended sense of 'theory').

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- *Intuitively*: two logical systems are notational variants if and only if, once we translate them properly, they give rise to theories that are coextensive modulo translation (in our extended sense of 'theory').

- (Notice that the coextensiveness modulo translation of $\mathbf{L}_1$ and $\mathbf{L}_2$ *alone* follows from the special cases where $\mathfrak{I}$ or $\mathfrak{S}$ is $\varnothing$.)

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The Non-Transitive Challenge

- It is trivial to check that **CL** and **ST** aren't the same under (IA).
- It is also easy to check that the paradigmatic cases of notational variance (same system with different symbols, primitive constants, etc.) are respected.
- Thus, I take (IA) to be significant improvement over (EAD).

Plan

1. The Standard Approach
2. The Non-Reflexive Challenge
3. The Non-Transitive Challenge
4. Taking Stock

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- Of course, it remains to be determined whether (IA) is entirely satisfactory. One possible objection could be, for instance, that it cannot distinguish between $\mathbf{ST}_1$, $\mathbf{ST}_2$,... and $\mathbf{ST}_\omega$.

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- I analyzed how this applies, in particular, to our extant criteria of notational variance. I claimed that those criteria don't live up to our expectations, and offered a new one.
- Of course, it remains to be determined whether (IA) is entirely satisfactory. One possible objection could be, for instance, that it cannot distinguish between $\mathbf{ST}_1$, $\mathbf{ST}_2$,... and $\mathbf{ST}_\omega$.
- But I don't know how serious this criticism is—given the unclear function of high meta$_n$inferences. (Perhaps one step is enough—Ripley [21].)

References I

Eduardo Barrio, Lucas Rosenblatt, and Diego Tajer.
The logics of strict-tolerant logic.
Journal of Philosophical Logic, 44(5):551–571, 2015.

Eduardo Alejandro Barrio, Federico Pailos, and Damian Szmuc.
A hierarchy of classical and paraconsistent logics.
Journal of Philosophical Logic, 49(1), 2020.

Eduardo Alejandro Barrio, Federico Pailos, and Damian Szmuc.
Substructural logics, pluralism and collapse.
Synthese, 198(20):4991–5007, 2021.

Nuel D Belnap.
Tonk, plonk and plink.
Analysis, 22(6):130–134, 1962.

References II

Carlos Caleiro and Ricardo Gonçalves.
Equipollent logical systems.
In *Logica Universalis: Towards a general theory of logic*, pages 97–109. Springer, 2007.

P. Cobreros, P. Egré, D. Ripley, and R. Van Rooij.
Reaching transparent truth.
Mind, 122(488):841–866, 2013.

Pablo Cobreros, Paul Egré, David Ripley, and Robert van Rooij.
Inferences and metainferences in st.
Journal of Philosophical Logic, 49(6):1057–1077, 2020.

Fabrice Correia.
Logical grounds.
The review of symbolic logic, 7(1):31–59, 2014.

References III

Neil Dewar.
On translating between logics.
Analysis, 78(4):622–630, 2018.

Bogdan Dicher and Francesco Paoli.
St, lp and tolerant metainferences.
In *Graham Priest on dialetheism and paraconsistency*, pages 383–407. Springer, 2019.

Melvin Fitting.
The strict/tolerant idea and bilattices.
In *Arnon Avron on Semantics and Proof Theory of Non-Classical Logics*, pages 167–191. Springer, 2021.

Rohan French.
Structural reflexivity and the paradoxes of self-reference.
Ergo, an Open Access Journal of Philosophy, 3, 2016.

References IV

Rohan French.
Notational variance and its variants.
Topoi, 38:321–331, 2019.

Lloyd Humberstone.
Heterogeneous logic.
Erkenntnis, 29(3):395–435, 1988.

Alexander W Kocurek.
On the concept of a notational variant.
In *Logic, Rationality, and Interaction*, pages 284–298. Springer, 2017.

Carlo Nicolai and Lorenzo Rossi.
Systems for non-reflexive consequence.
Studia Logica, pages 1–31, 2023.

References V

Federico Matias Pailos.
A fully classical truth theory characterized by substructural means.
The Review of Symbolic Logic, 13(2):249–268, 2020.

Francis Jeffry Pelletier and Alasdair Urquhart.
Synonymous logics.
Journal of Philosophical Logic, 32:259–285, 2003.

Brian Porter.
Supervaluations and the strict-tolerant hierarchy.
Journal of Philosophical Logic, pages 1–20, 2021.

David Ripley.
Conservatively extending classical logic with transparent truth.
Review of Symbolic Logic, 5(02):354–378, 2012.

References VI

David Ripley.
One step is enough.
Journal of Philosophical Logic, pages 1–27, 2021.

Chris Scambler.
Classical logic and the strict tolerant hierarchy.
Journal of Philosophical Logic, 49(2):351–370, 2020.

Damian Szmuc.
Non-transitive counterparts of every tarskian logic.
Analysis, 2023.

John Wigglesworth.
Logical anti-exceptionalism and theoretical equivalence.
Analysis, 77(4):759–767, 2017.

References VII

Timothy Williamson.
Semantic paradoxes and abductive methodology.
Reflections on the Liar, pages 325–346, 2017.

Jack Woods.
Intertranslatability, theoretical equivalence, and perversion.
Thought: A Journal of Philosophy, 7(1):58–68, 2018.

Thanks!!