Epistemic Skills

Logical Dynamics of Knowing and Forgetting

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Similarity between Compound Data

Name	Age	Hobbies
Alice	25	reading, hiking, cooking
Alicia	27	reading, dancing, cooking

- Jaccard similarity for hobbies (sets):

H_1 = {“reading”, “hiking”, “cooking”}
H_2 = {“reading”, “dancing”, “cooking”}

$$\frac{|H_1 \cap H_2|}{|H_1 \cup H_2|} = \frac{2}{4} = 0.5$$

- Jaro-Winkler similarity for names:

$$sim_{jw}(\text{“Alice”}, \text{“Alicia”}) = 0.893$$

- Numeric similarity for ages:

$$1 - \frac{|25 - 27|}{\max(25, 27)} = 1 - \frac{2}{27} \approx 0.926$$

- Composite similarity:

$$\frac{0.893 + 0.926 + 0.5}{3} = 0.773$$

Gene	Sample ID	Expression Level (FPKM)	P-Value	Log Fold Change
BRCA1	Sample001	35.4	0.002	2.3
TP53	Sample001	50.2	0.005	-1.8
HER2	Sample002	25.1	0.001	3.1
EGFR	Sample002	45.6	0.003	-2.0

Retention Time (min)	Peak Area	Compound	Concentration (mg/L)
5.2	1500	Compound A	12.5
10.6	2500	Compound B	25.0
15.3	3000	Compound C	30.5

Chemical Shift (δ ppm)	Multiplicity	Coupling Constant (Hz)
1.25	Singlet	-
3.45	Doublet	7.2
7.60	Triplet	15.3

Cell Population	Marker	Fluorescence Intensity (MFI)
CD4+ T cells	CD25	150
CD8+ T cells	CD69	200
B cells	CD19	100

Abstract Similarity between Complex Worlds

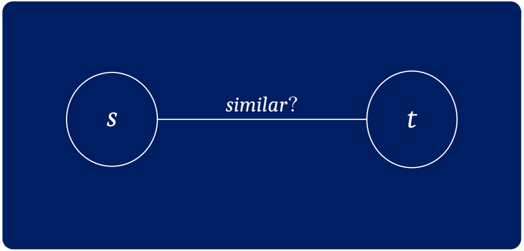
A generalization from similarity degrees to aspects

S: a set of skills/aspects

S1	$s(x, y) \subseteq S$	(abstract measure)
S2a	$s(x, y) = S \implies x = y$	(congruence implies equality)
S3	$s(x, y) = s(y, x)$	(symmetry)

From similarity to knowledge

similarity
between possible worlds
$\Downarrow$
indistinguishability
of possible worlds
$\Downarrow$
uncertainty
$\Downarrow$
knowledge:
propositions free from uncertainty



Models

We have chosen a general way of representing similarity

P: atoms

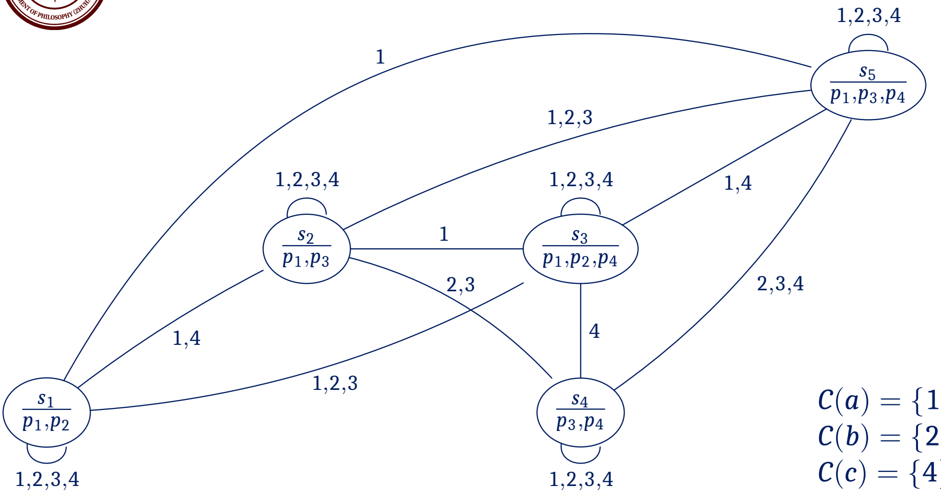
A: agents

S: epistemic skills

A model is a quadruple (W, E, C, V):

- W: worlds / states / nodes
- $E : W \times W \to \wp(S)$: edge function
- $C : A \to \wp(S)$: capability function
- $V : W \to \wp(P)$: valuation

Illustration of a Model



$$C(a) = \{1, 2, 3\}$$
$$C(b) = \{2, 3, 4\}$$
$$C(c) = \{4\}$$

Basic Language and Formal Semantics

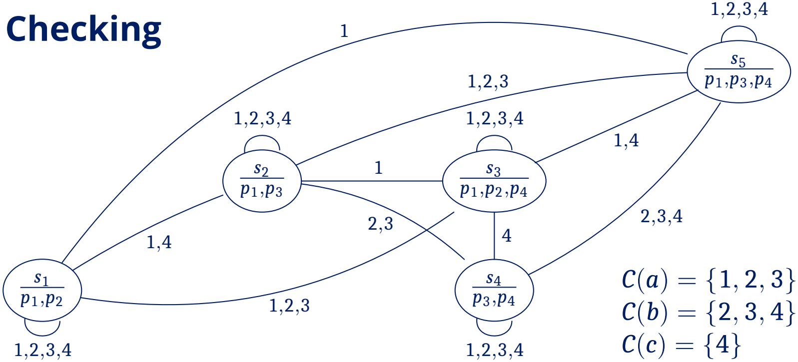
Epistemic language: $\phi ::= p \mid \neg\phi \mid (\phi \to \phi) \mid K_a\phi$

Satisfaction:

$$
\begin{aligned}
M, s \models p &\iff p \in \nu(s) \\
M, s \models \neg\psi &\iff \text{not } M, s \models \psi \\
M, s \models \psi \to \chi &\iff \text{if } M, s \models \psi \text{ then } M, s \models \chi \\
M, s \models K_a\phi &\iff \text{for all } t \in W, \text{ if } C(a) \subseteq E(s,t) \text{ then } M, t \models \phi
\end{aligned}
$$

- $C(a)$: agent a's skill set
- $E(s,t)$: skills with which one cannot discern between s and t
- $C(a) \subseteq E(s,t)$: a **cannot discern between** s **and** t

Model Checking



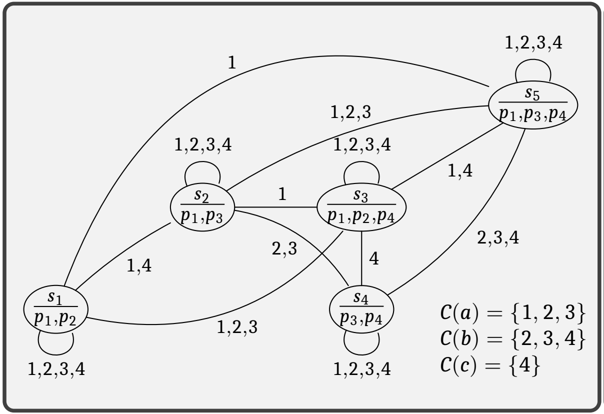
$C(a) = \{1, 2, 3\}$
$C(b) = \{2, 3, 4\}$
$C(c) = \{4\}$

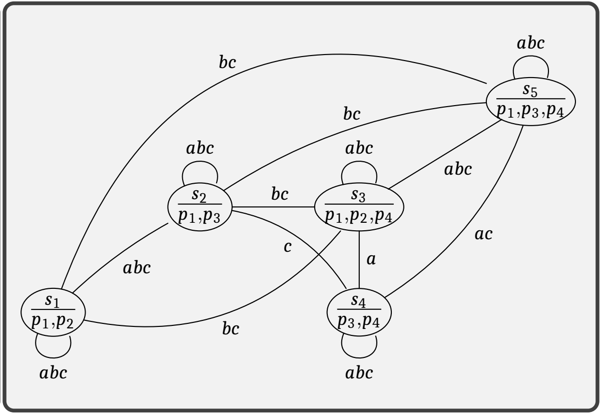
$s_2 \models K_a p_3$

$s_4 \models \neg K_b p_1 \wedge \neg K_b \neg p_1$

$s_3 \models K_c(K_a p_3 \vee K_a \neg p_3)$

Translation to Classical Kripke Model





Some Results

- Complexity of model checking: in P
- Satisfiability/validity problem: PSPACE complete
- Axiomatization: **KB**

Liang X. & Wáng, Y.N. Epistemic Logics over Weighted Graphs. LNGAI 2022.

Similarity Metrics (CMZ2009) *

We can also go for more concrete similarity measures

S1′	$s(x,x) \geq 0$	(nonnegative self-similarity)
S2a′	$s(x,y) = 1 \Longrightarrow x = y$	(congruence implies equality)
S2b	$s(x,x) \geq s(x,y)$	(high self-similarity)
S3	$s(x,y) = s(y,x)$	(symmetry)
S4	$s(x,z) \geq s(x,y) + s(y,z) - s(y,y)$	(sharp triangularity)

Liang X. & Wáng, Y.N. Epistemic Logics via Distance and Similarity. PRICAI 2022.
Liang X. & Wáng, Y.N. Similarity Metrics from the Perspective of Epistemic Logic. manuscript.

Incorporating Group Knowledge

CK, DK, EK and FK

Notions of Group Knowledge

- Individual knowledge: $K_a \phi$
- Mutual/Everyone's knowledge: $E_G \phi := \bigwedge_{x \in G} K_x \phi$
- Common knowledge: $C_G \phi$, make sure that $\models C_G \phi \leftrightarrow E_G(\phi \wedge C_G \phi)$
- Distributed knowledge: $D_G \phi$, to be reinterpreted
- Field knowledge: $F_G \phi$, new

Liang X. & Wáng, Y.N. Field Knowledge as a Dual to Distributed Knowledge: A Characterization by Weighted Modal Logic. LNGAI 2024.

Semantics

Model $M = (W, E, C, V)$

$$M, s \models K_a\psi \iff \text{for all } t \in W, \text{ if } C(a) \subseteq E(s,t) \text{ then } M, t \models \psi$$
$$M, s \models E_G\psi \iff \text{for all } a \in G, M, s \models K_a\psi$$
$$M, s \models C_G\psi \iff \text{for all } n \in \mathbb{N}^+, M, s \models E_G^n\psi$$

$$M, s \models D_G\psi \iff \text{for all } t \in W, \text{ if } \bigcup_{a\in G} C(a) \subseteq E(s,t) \text{ then } M, t \models \psi$$
$$M, s \models F_G\psi \iff \text{for all } t \in W, \text{ if } \bigcap_{a\in G} C(a) \subseteq E(s,t) \text{ then } M, t \models \psi$$

- Distributed knowledge: the group's knowledge by combing the individual skills
- Field knowledge: the group's knowledge by their common skills

Semantics

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- Distributed knowledge: the group's knowledge by combing the individual skills
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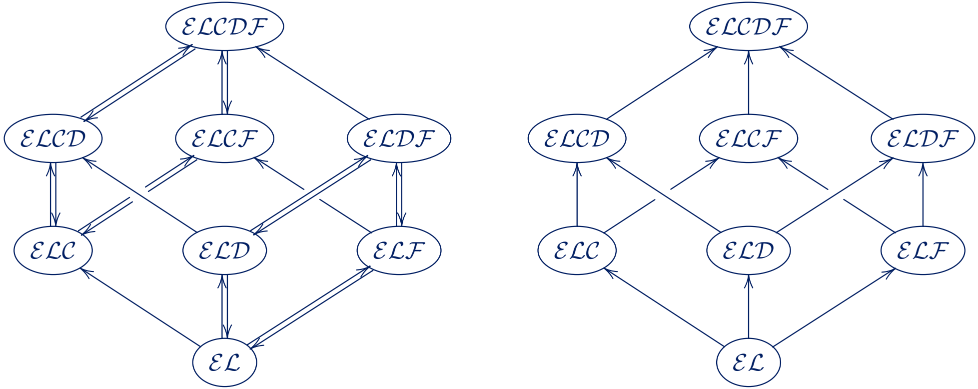
Compare:

- $M, s \models E_G\psi \iff$ for all $t \in W$, if $(s,t) \in \bigcup_{a\in G} R_a$, then $M, t \models \psi$
- $M, s \models D_G\psi \iff$ for all $t \in W$, if $(s,t) \in \bigcap_{a\in G} R_a$, then $M, t \models \psi$

Expressivity *



(a) when $|Ag| = 1$ **(b)** when $|Ag| \geq 2$

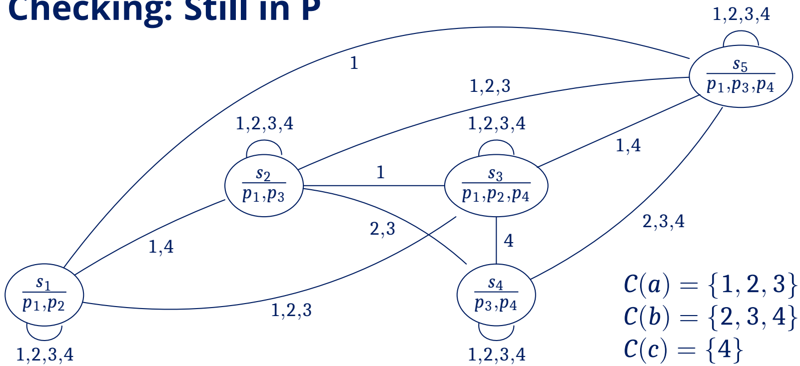
Axiomatization *

- Base system: $\mathbf{KB}$
- System $\mathbf{F}$
  - (K$_F$) $F_G(\phi \to \psi) \to (F_G\phi \to F_G\psi)$
  - (F1) $F_{\{a\}}\phi \leftrightarrow K_a\phi$
  - (F2) $F_G\phi \to F_H\phi$ with $H \subseteq G$
  - (BF) $\phi \to F_G\neg F_G\neg\phi$
  - (NF) from ϕ infer $F_G\phi$
- System $\mathbf{C}$
  - (C1) $C_G\phi \to \bigwedge_{a\in G} K_a(\phi \wedge C_G\phi)$
  - (C2) from $\phi \to \bigwedge_{a\in G} K_a(\phi \wedge \psi)$ infer $\phi \to C_G\psi$
- System $\mathbf{D}$
  - (K$_D$) $D_G(\phi \to \psi) \to (D_G\phi \to D_G\psi)$
  - (D1) $D_{\{a\}}\phi \leftrightarrow K_a\phi$
  - (D2) $D_G\phi \to D_H\phi$ with $G \subseteq H$
  - (BD) $\phi \to D_G\neg D_G\neg\phi$

Completeness proofs *

- By translation of satisfiability
  - $\mathbf{KB}$
- Canonical model method
  - $\mathbf{KB}$
- Path-based canonical models (unraveling/folding)
  - $\mathbf{KB} \oplus \mathbf{D}$, $\mathbf{KB} \oplus \mathbf{F}$, $\mathbf{KB} \oplus \mathbf{D} \oplus \mathbf{F}$
- Finitary path-based canonical models
  - $\mathbf{KB} \oplus \mathbf{C}$, $\mathbf{KB} \oplus \mathbf{C} \oplus \mathbf{D}$, $\mathbf{KB} \oplus \mathbf{C} \oplus \mathbf{F}$, $\mathbf{KB} \oplus \mathbf{C} \oplus \mathbf{D} \oplus \mathbf{F}$

Model Checking: Still in P



$C(a) = \{1, 2, 3\}$
$C(b) = \{2, 3, 4\}$
$C(c) = \{4\}$

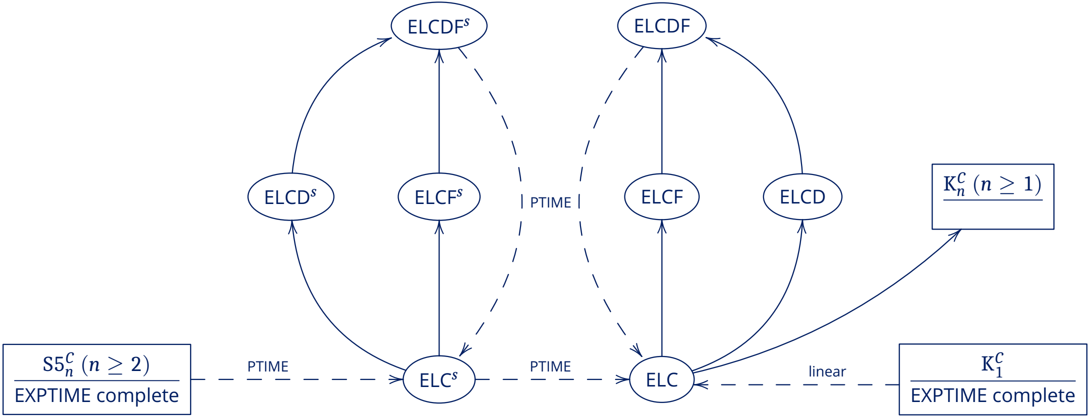
$s_2 \models K_a p_3$

$s_4 \models \neg F_{\{a,b\}} \neg p_1$

$s_5 \models \neg C_{\{a,c\}} p_1$

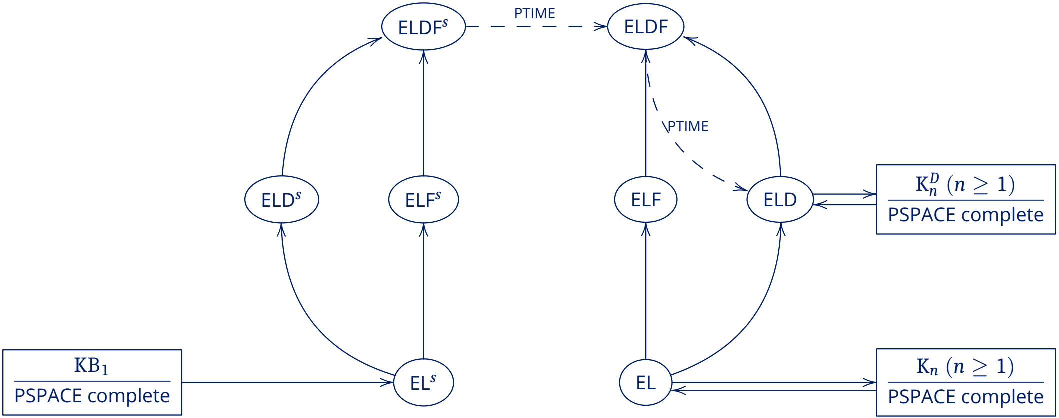
Computational complexity of SAT

Logics with CK: EXPTIME complete



Computational complexity of SAT

Logics without CK: PSPACE complete



Dynamics

Knowing and forgetting

Upskilling, Downskilling and Reskilling

$$\phi \quad ::= \quad p \mid \neg\phi \mid (\phi \to \phi) \mid K_a\phi \mid C_G\phi \mid D_G\phi \mid E_G\phi \mid F_G\phi \mid$$
$$(+_S)_a\phi \mid (-_S)_a\phi \mid (=_S)_a\phi \mid (\equiv_b)_a\phi \mid \boxplus_a\phi \mid \boxminus_a\phi \mid \Box_a\phi$$

$M, w \models (+_S)_a\psi \Leftrightarrow W, E, C^{a+S}, \beta, w \models \psi$ $\quad C^{a+S}(a)=C(a)\cup S$ and $\forall x \in A\backslash\{a\}.\ C^{a+S}(x)=C(x)$

$M, w \models (-_S)_a\psi \Leftrightarrow W, E, C^{a-S}, \beta, w \models \psi$ $\quad C^{a-S}(a)=C(a)\backslash S$ and $\forall x \in A\backslash\{a\}.\ C^{a-S}(x)=C(x)$

$M, w \models (=_S)_a\psi \Leftrightarrow W, E, C^{a=S}, \beta, w \models \psi$ $\quad C^{a=S}(a)=S$ and $\forall x \in A\backslash\{a\}.\ C^{a=S}(x)=C(x)$

$M, w \models (\equiv_b)_a\psi \Leftrightarrow W, E, C^{a\equiv b}, \beta, w \models \psi$ $\quad C^{a\equiv b}(a)=C(b)$ and $\forall x \in A\backslash\{a\}.\ C^{a\equiv b}(x)=C(x)$

$M, w \models \boxplus_a\psi \quad \Leftrightarrow$ for all finite nonempty $S \subseteq \mathrm{S}$, $M, w \models (+_S)_a\psi$

$M, w \models \boxminus_a\psi \quad \Leftrightarrow$ for all finite nonempty $S \subseteq \mathrm{S}$, $M, w \models (-_S)_a\psi$

$M, w \models \Box_a\psi \quad \Leftrightarrow$ for all finite nonempty $S \subseteq \mathrm{S}$, $M, w \models (=_S)_a\psi$

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Slogans

Forgetting: decrease in skills, and increase in uncertainty

Necessary: true in all accessible worlds.
Known: true in all uncertain situations.

APAL:"Knowable as known after an announcement."

Slogan 1. Knowable as known after upskilling.
Slogan 2. Forgettable as unknown after downskilling.

Debate: having no access is not forgetting.

Epistemic De Re & De Dicto

Von Wright (1951) *An Essay in Modal Logic*

- The epistemic modalities are said to be de dicto when they are about the mode or way in which a proposition is or is not known (to be true). The epistemic modalities are used de dicto in phrases such as "it is known that ...", "it is unknown whether ...", or "it is known that not ...".
- The epistemic modalities are said to be de re when they are about the mode or way in which an individual thing is known to possess or to lack a certain property. The modalities are used de re in phrases such as "Jones is (not) known (not) to be dead", etc.

Epistemic De Re & De Dicto

Quine (1956) *Quantifiers and Propositional Attitudes*

- "Ralph believes that someone is a spy."
  - Ralph believes that there is a spy.
    Ralph believes: $\exists x (x \text{ is a spy})$.
  - Someone is such that Ralph believes that s/he is a spy.
    $\exists x$ (Ralph believes that x is a spy).
- Ambiguity comes from the scope of the quantifier

Knowing De Dicto in our case

- "Agent a knows (with her current skills) that there exists a set S of skills such that, with S, she can achieve ϕ in world w of model (W, E, C, β)."
- $(\forall u \in W)[C(a) \subseteq E(w, u) \Rightarrow (\exists S \subseteq \mathsf{S})\ (W, E, C^{a+S}, \beta), u \models \phi]$
- Expressed by $K_a \oplus_a \phi$

Knowing De Re in our case

- (Explicitly knowing de re) There exists a set S of skills such that agent a knows with her current skill set, that with S in addition, she can achieve ϕ in world w of model (W, E, C, β).
  $(\exists S \subseteq \mathsf{S})(\forall u \in W)[C(a) \subseteq E(w, u) \Rightarrow (W, E, C^{a+S}, \beta), u \models \phi]$
  Expressed by $(\equiv_a)_c \oplus_c K_a (\equiv_c)_a \phi$ (where c is not in ϕ)
- (Implicitly knowing de re) There exists a set S of skills such that agent a knows, with the addition of S to her skill set, that she can achieve ϕ in world w of model (W, E, C, β).
  $(\exists S \subseteq \mathsf{S})(\forall u \in W)[C^{a+S}(a) \subseteq E(w, u) \Rightarrow (W, E, C^{a+S}, \beta), u \models \phi]$
  Expressed by $\oplus_a K_a \phi$

- Logics without quantifiers: in P
- Logics with quantifiers: PSPACE complete
  - Hardness: reducing the Undirected Edge Geography (UEG) problem

Upper Bound

We only need to consider one new skill in addition to those that already appear

Algorithm Function $Val((W, E, C, \beta), \varphi)$:

Initialize: $temVal \leftarrow \emptyset$

Initialize: $S_1 \leftarrow (\bigcup_{w,v \in W} E(w,v)) \cup (\bigcup_{a \text{ appears in } \varphi} C(a))$

Initialize: $S_2 \leftarrow S_1 \cup \{s\}$ ▷ *Here* $s \in S$ *is new for* S_1

if ... **then** ...

else if $\varphi = \boxplus_a \psi$ **then**

  for all $t \in W$ **do**

    Initialize: $n \leftarrow$ **true**

    for all $S \subseteq S_2$ **do**

      if $S \neq \emptyset$ **and** $t \notin Val((W, E, C^{a+S}, \beta), \psi)$ **then** $n \leftarrow$ **false**

    if $n =$ **true then** $tmpVal \leftarrow tmpVal \cup \{t\}$

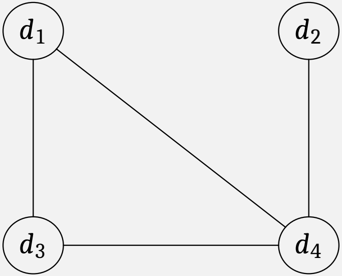
  return $tmpVal$

▷ *Returns* $\{t \in W \mid \forall S \subseteq S_1 : t \in Val((W, E, C^{a+S}, \beta), \psi)\}$

else if ... **then** ...

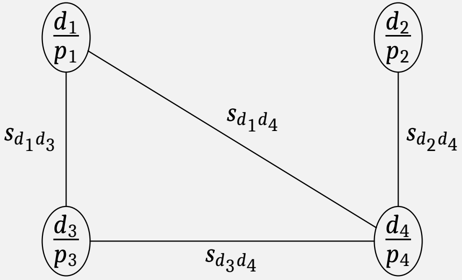
Example: UEG Game on (G, d_1)

$$G = \Big(\{d_1, d_2, d_3, d_4\}, \{(d_1, d_3), (d_1, d_4), (d_2, d_4), (d_3, d_4)\}\Big)$$



Model $M_G = (W, E, C, \beta)$

$W = \{d_1, \ldots, d_4\}$



$C(a_1) = \emptyset$
$C(a_2) = \emptyset$
$C(a_3) = \emptyset$
$C(a_4) = \emptyset$

- $E(d_m, d_k) = \{s_{d_m d_k}\}$ whenever d_m —— d_k
- $C(a_1) = C(a_2) = C(a_3) = C(a_4) = \emptyset$ (a_i is the player who performs the i's move)
- $V(d_j) = \{p_j\}$ for $1 \leq j \leq 4$

Formula ϕ_G

For *i*'s move in the UEG game:

$$\psi_i := \neg K_{a_i}\bot \wedge (K_{a_i}p_1 \vee K_{a_i}p_2 \vee K_{a_i}p_3 \vee K_{a_i}p_4)$$

$$\chi_1 := \bot$$

$$\chi_2 := (\hat{K}_{a_1}p_1 \wedge K_{a_2}p_1) \vee (\hat{K}_{a_1}p_2 \wedge K_{a_2}p_2) \vee (\hat{K}_{a_1}p_3 \wedge K_{a_2}p_3) \vee (\hat{K}_{a_1}p_4 \wedge K_{a_2}p_4)$$

$$\begin{aligned}\chi_3 := {} & (\hat{K}_{a_1}p_1 \wedge K_{a_2}p_1) \vee (\hat{K}_{a_1}p_2 \wedge K_{a_2}p_2) \vee (\hat{K}_{a_1}p_3 \wedge K_{a_2}p_3) \vee (\hat{K}_{a_1}p_4 \wedge K_{a_2}p_4) \\ & \vee(\hat{K}_{a_1}p_1 \wedge K_{a_3}p_1) \vee (\hat{K}_{a_1}p_2 \wedge K_{a_3}p_2) \vee (\hat{K}_{a_1}p_3 \wedge K_{a_3}p_3) \vee (\hat{K}_{a_1}p_4 \wedge K_{a_3}p_4) \\ & \vee(\hat{K}_{a_2}p_1 \wedge K_{a_3}p_1) \vee (\hat{K}_{a_2}p_2 \wedge K_{a_3}p_2) \vee (\hat{K}_{a_2}p_3 \wedge K_{a_3}p_3) \vee (\hat{K}_{a_2}p_4 \wedge K_{a_3}p_4)\end{aligned}$$

$$\chi_i := \bigvee_{1\leq j<i} ((\hat{K}_{a_j}p_1 \wedge K_{a_i}p_1) \vee (\hat{K}_{a_j}p_2 \wedge K_{a_i}p_2) \vee (\hat{K}_{a_j}p_3 \wedge K_{a_i}p_3) \vee (\hat{K}_{a_j}p_4 \wedge K_{a_i}p_4))$$

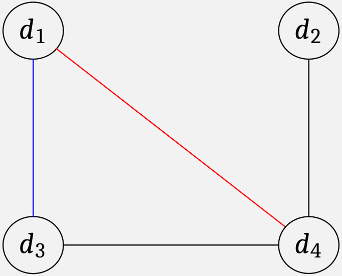
$$\phi_G := \diamondplus_{a_1}(\psi_1 \wedge \neg\chi_1 \wedge K_{a_1} \boxplus_{a_2} (\neg\psi_2 \vee \chi_2 \vee \hat{K}_{a_2}\diamondplus_{a_3}(\psi_3 \wedge \neg\chi_3 \wedge K_{a_3} \boxplus_{a_4} (\neg\psi_4 \vee \chi_4))))$$

The following are equivalent

- Player 1 has a winning strategy in (G, d_1)
- $M_G, d_1 \models \phi_G$

Player 1's Move for Step 1

$$G = \Big(\{d_1, d_2, d_3, d_4\}, \{(d_1, d_3), (d_1, d_4), (d_2, d_4), (d_3, d_4)\}\Big)$$



- Player 1 chooses blue: will win
- Player 1 chooses red: can loose

First Step in the Model Checking

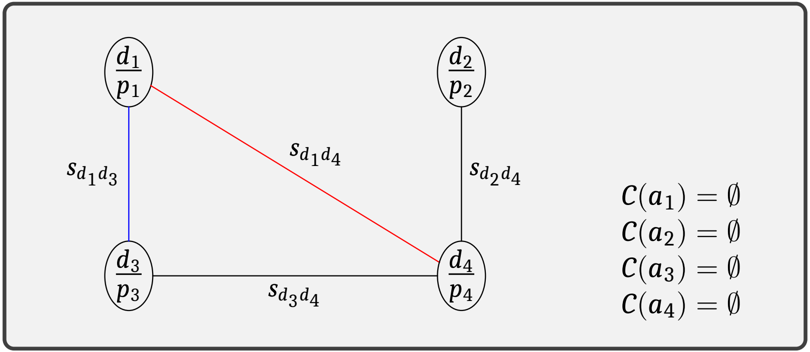
$M_G, d_1 \models \phi_G$, where ϕ_G is:

$$\diamondplus_{a_1}\left(\psi_1 \wedge \neg\chi_1 \wedge K_{a_1} \boxplus_{a_2} (\neg\psi_2 \vee \chi_2 \vee \hat{K}_{a_2} \diamondplus_{a_3}(\psi_3 \wedge \neg\chi_3 \wedge K_{a_3} \boxplus_{a_4} (\neg\psi_4 \vee \chi_4)))\right)$$

After some upskilling for a_1, true in d_1 are:

- $\psi_1 = \neg K_{a_1}\bot \wedge (K_{a_1}p_1 \vee K_{a_1}p_2 \vee K_{a_1}p_3 \vee K_{a_1}p_4)$
- $\neg\chi_1 = \neg\bot$
- $K_{a_1} \boxplus_{a_2} (\neg\psi_2 \vee \chi_2 \vee \hat{K}_{a_2} \diamondplus_{a_3}(\psi_3 \wedge \neg\chi_3 \wedge K_{a_3} \boxplus_{a_4} (\neg\psi_4 \vee \chi_4)))$

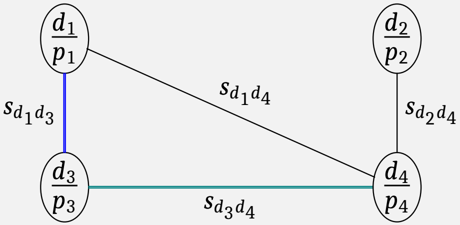
Step 1: Model Checking



$$M_G, d_1 \models (+\{s_{d_1d_3}\})_{a_1}\left(\psi_1 \wedge \neg\chi_1 \wedge K_{a_1} \boxplus_{a_2} (\neg\psi_2 \vee \chi_2 \vee \hat{K}_{a_2} \oplus_{a_3} (\psi_3 \wedge \neg\chi_3 \wedge K_{a_3} \boxplus_{a_4} (\neg\psi_4 \vee \chi_4)))\right)$$

$$M_G, d_1 \not\models (+\{s_{d_1d_4}\})_{a_1}\left(\psi_1 \wedge \neg\chi_1 \wedge K_{a_1} \boxplus_{a_2} (\neg\psi_2 \vee \chi_2 \vee \hat{K}_{a_2} \oplus_{a_3} (\psi_3 \wedge \neg\chi_3 \wedge K_{a_3} \boxplus_{a_4} (\neg\psi_4 \vee \chi_4)))\right)$$

Step 2: Blue Case



$C(a_1) = \{s_{d_1d_3}\}$
$C(a_2) = \emptyset$
$C(a_3) = \emptyset$
$C(a_4) = \emptyset$

$M_G, d_3 \models \boxplus_{a_2}(\neg\psi_2 \vee \chi_2 \vee \hat{K}_{a_2} \oplus_{a_3}(\psi_3 \wedge \neg\chi_3 \wedge K_{a_3} \boxplus_{a_4}(\neg\psi_4 \vee \chi_4)))$

- $M_G, d_3 \models (+\{s_{d_1d_3}\})_{a_2}(\neg\psi_2 \vee \chi_2 \vee \hat{K}_{a_2} \oplus_{a_3}(\psi_3 \wedge \neg\chi_3 \wedge K_{a_3} \boxplus_{a_4}(\neg\psi_4 \vee \chi_4)))$
- $M_G, d_3 \models (+\{s_{d_3d_4}\})_{a_2}(\neg\psi_2 \vee \chi_2 \vee \hat{K}_{a_2} \oplus_{a_3}(\psi_3 \wedge \neg\chi_3 \wedge K_{a_3} \boxplus_{a_4}(\neg\psi_4 \vee \chi_4)))$
- $M_G, d_3 \models (+\{s_{d_1d_4}\})_{a_2}(\neg\psi_2 \vee \chi_2 \vee \hat{K}_{a_2} \oplus_{a_3}(\psi_3 \wedge \neg\chi_3 \wedge K_{a_3} \boxplus_{a_4}(\neg\psi_4 \vee \chi_4)))$ (or any other combinations)

Step 2: Red Case

$C(a_1) = \{s_{d_1 d_4}\}$
$C(a_2) = \emptyset$
$C(a_3) = \emptyset$
$C(a_4) = \emptyset$

$M_G, d_4 \not\models \boxplus_{a_2}(\neg\psi_2 \vee \chi_2 \vee \hat{K}_{a_2} \diamondplus_{a_3}(\psi_3 \wedge \neg\chi_3 \wedge K_{a_3} \boxplus_{a_4}(\neg\psi_4 \vee \chi_4)))$

- $M_G, d_4 \not\models (+\{s_{d_2 d_4}\})_{a_2}(\neg\psi_2 \vee \chi_2 \vee \hat{K}_{a_2} \diamondplus_{a_3}(\psi_3 \wedge \neg\chi_3 \wedge K_{a_3} \boxplus_{a_4}(\neg\psi_4 \vee \chi_4)))$

Future Work

- Computational complexity of the satisfiability/validity problems
- Logics over different characterizations of similarity
- Axiomatization

Thank you!